CHAPTER I

ON THE THEORY OF THE WEIERSTRAß $\wp$-FUNCTION AND OF THE ELLIPTIC FUNCTIONS.

I. THE WEIERSTRAß $\wp$-FUNCTION.

The theory of the Weierstraß $\wp$-function embodies precious instruments for the improvement of the mathematical methods of the Theory of General Relativity, which I outline in the present work.

The theory of the Weierstraß $\wp$-function has been neglected since the early Twenties of 20th Century and only few not comprehensive works have been published thereafter.

This Chapter provides a comprehensive insight into its theory.

Its first part is based on Volume V and VI of the Weierstraß's Mathematical Works and of the Notes to his Lessons on the Theory of the Elliptic Functions held in Winter-Semester 1882-1883 at the Berlin University.

The successive part of the Chapter concerns a short outline of the distinct ways some mathematicians introduced the Weierstraß elliptic functions, and the main properties of and formulae for the Weierstraß $\wp$-function, the Jacobian elliptic functions, the first kind elliptic integral, the connections between geometry and the Weierstraß elliptic functions, and the form of the real part of and of the coefficient of the imaginary part of the Weierstraß $\wp$-function.

Historical introduction.

Weierstraß introduced the theory of the ℘-function in his Courses held at the University of Berlin more than one Century ago, and he developed it throughout the second half of the 19th Century.

There exist many ways to define and prove the properties and theorems on and formulae for the Weierstraß ℘-function, but the most complete one is the 'historical' way, i.e. the outline of the results outlined by Weierstraß himself, joined with some other successive results outlined in the most important treatises thereon, on a chronological basis.

I observe that the goal to publish a complete treatise on the Weierstraß elliptic functions and ℘-function can be achieved only by 'deciphering' the Kronecker form of the Theory of Elliptic Functions.

I observe that its common knowledge that such a task is yet to be accomplished(6).

There are no comprehensive works devoted to the ℘-function published directly by Weierstraß, for he never wrote down a complete treatise and he was averse to the publication of the notes to his Lectures, but there are handwritten notes or copies of handwritten notes to his Lectures(7).

Volume V, that outlines the very Weierstraß's theory of the elliptic functions and of the ℘-function, and VI, that concerns many applications of the elliptic functions and of the ℘-function, and many parts of which are extensions and developments of some topics outlined in Volume V, of the Weierstraß's Mathematical Works(8) are based on some among these handwritten notes.

Volume V is based on handwritten notes by Mertens of the course of Lectures of year 1863 and on a revision by Felix Müller of notes to the course of Lectures of Winter Semester 1864-65.

Nevertheless, the application of the results outlined by Volume V shows an intrinsic limit, for the handwritten notes, which it is based on, are missing(9).

Volume VI is based on revisions by Hettner of the Lectures of summer 1875, on not well-specified notes in possession of the Berliner Mathematisches Verein, on a copy of Weierstraß's notes of year 1884, on a

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(6) The publications of many works on the various aspects of the Kronecker's mathematical researches did not lead to a real moving toward this goal. The greatest part of them are substantially low-brand works showing only the presumption of their authors. Only a few can be considered worth of consideration.

(7) The Weierstraß's Lectures are the only rigorous source. The best treatises on the Weierstraß ℘-function and on the Weierstraß elliptic functions are strictly based on them. Some European archives hold many handwritten notes to his Lectures. I refer the reader to my works *‚Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil I ('Die im Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin beibehaltenen Akten der Kommission für die Herausgabe der Werke von Weierstrass und das heutige Schicksal der die an der Universität Berlin von Weierstrass angehaltenen Vorlesungen betreffenden Manuskripte'), Teil II (‚Der wichtigste Teil des vom Geheimen Staatsarchive Preußischen Kulturbesitze in Berlin-Dahlem beibehaltenen Nachlasses von Karl Wilhelm Theodor Weierstrass'), Teil III (‚Der bei der Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin behaltene Schwarz-Weierstrasssche Briefwechsel und die bei dem Archive der Universität zu Göttingen behaltene ausgewählte Weierstrasssche Korrespondenz')',* on display on Google Play and Google Books. Particularly, I consider the actual fate of the Notes to the Weierstraß's Lectures in the Teil I of my works. It is self-evident that the publication and analysis of the whole corpus of the notes based on Weierstraß's Lectures is the only way to set up a rigorous outline of Weierstraß's mathematical thinking. However, the evident amount of space and time needed to achieve this goal does not allow me carrying out this analysis in this work. This work should be considered the real Weierstraßiana, i.e. the effective 'Complete Mathematical Works' by Weierstraß.

(8) See Weierstraß [III] and [V].

(9) I refer the reader to S. 123-125 of Teil I of my above-mentioned works.

Weierstraß's manuscript of year 1888 in possession of Sophie von Kowalewski, on notes by Weltzien to a Lecture held in summer 1873, on a revision by Felix Müller to notes to some Lectures held in summer 1865, on a revision by Kiepert to notes to some Lectures of year 1869 and on a Chapter of a Schellbach's work(10).

There exist fundamental differences between the results outlined by Schellbach and the Weierstraß's method, for Schellbach outlines 'de facto' a synoptic Table of transformations(11), while Weierstraß presents a complete outline of the method.

A possible explication is that the Weierstraß's Lecture on the elliptic functions mentioned by Schellbach is not among the Lectures on which Volume V of his Mathematical Works is based.

I could not check the above-mentioned manuscripts.

Therefore, I cannot state what kind – if any – of additional notes and corrections they show.

However, I could read the notes to the Lectures by Weierstraß on the application of the elliptic functions held in Winter-Semester 1882-1883(12).

Some parts of works published in the other volumes of the Weierstraß's Mathematical Works(13) show some results connected with the theory of the $\wp$-function.

I outline the English translations of the parts of these works connected with the main body of the mentioned Weierstraß' works.

The only other published work on the Weierstraß elliptic functions directly based on some manuscripts written and revised directly by Weierstraß is the well-known work by Schwarz(14).

The drafts of the work itself have been carefully checked by Weierstraß.

It is a project suggested by Schwarz to Weierstraß in order to publish an organic system of formulae for the Weierstraß elliptic functions, a need recognized implicitly by Weierstraß and supported by Schwarz mainly because the students at the Berlin University attending the Weierstraß's Lectures needed such an organic system of formulae to catch on with the results Weierstraß outlined steadily in his Lectures.

The Schwarz's work is – as well-known – a comprehensive outline of the basic definitions of, properties of and formulae for the Weierstraß elliptic functions and for the first, second and third kind elliptic integral. In addition, it shows some application of the Weierstraß elliptic function to the theory of conformal transformations.

Schwarz writes on S. VI of this work

"*I had the privilege of Weierstraß's help in writing this work, and no part of the first edition has been printed without his approval.*

The greatest part of the published formulae comes, as the title explains, from Lectures and handwritten notes by Weierstraß.

However, some formulae have been outlined in this work for the first time, because of their practical application. Kiepert outlined the formulae on the function

$$\frac{\sigma(nu)}{\sigma^{nn}(u)}$$

outlined in Art. 15, p. 19(15).

(10) See Schellbach [247], S. 258-275. Schellbach writes '*I wish to thank particularly Mr. Weierstraß for authorizing the use of some notes of the well-known master, for the thirteenth Section of the first Part is entirely based on them. … .*' and, on S. 259, '*This Section (A new method to reduce an elliptic differential to the canonical form) and his text have been outlined by Weierstraß in one of his Lectures on the elliptic function, and its publication has been authorized by him.*'.

(11) The Tables on the transformations of the elliptic integrals outlined by Magnus and Oberhettinger on pp. 132-138 of their work [3] are mainly based on the Schellbach's results, although almost all Schellbach's results hold if the coefficients have real values only and for values of x such that the fourth- or third-degree integral function $\mathcal{R}(x)$ has real values only.

(12) See Weierstraß [Ib]. I outline the translation of many parts of these notes connected with the topics of the Weierstraß's work [III].

(13) See Weierstraß [IV], [491] and [I].

(14) See Schwarz [II].

(15) *See Kiepert [175a], S. 31.*

I express my thankfulness to Hettner for the checking of the printed versions, the critics and observations during the various steps of the proofreading.

In addition, I wish to stress the accuracy and support of Hettner and Von Mangoldt in reviewing the published formulae.'.

The writing of this work spanned more than one decade.

Many information concerning the historical development of this work can be found in the correspondence between Schwarz and Weierstraß(16).

Daniels(17) published some remarks on this Schwarz's work in his work.

He writes(18)

"The work of Professor Weierstrass in the modern function-theory is of such commanding importance that it may not be out of place to give a clear and elementary account of his somewhat peculiar nomenclature and methods for the benefit of those English readers who have not had the opportunity of listening to his lectures.

This is especially desirable in the theory of doubly-periodic functions, where his symbols and methods differ not a little from those of Jacobi and his predecessors.

The only connected and systematic statement of Weierstrass' methods in this field is contained in the 'Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass bearbeitet und herausgegeben von H. A. Schwarz. Göttingen, 1882'.

Professor Schwarz has prepared this little work with infinite pains for the use of his own and Weierstrass' students exclusively.

I rely in the following chiefly on this work, on a large number of lithographed formulae prepared by Prof. Schwarz, and on notes of lectures on this topic by Profs. Schwarz and Weierstrass.(19)"

and he quotes some results, which can be directly ascribed to Weierstraß as follows(20)

'An analytic function of one variable cannot therefore have more than two classes of periods, because only two unit-pairs are possible so long as the fundamental laws of association and commutation hold. The Weierstraßian proof of this, which I venture to reproduce here, is as follows'.

I observe that Pringsheim(21) writes

"The work by Schwarz provides an exhaustive insight into the construction of the Theory."

Enneper(22) published a comprehensive treatise on the Weierstraß elliptic functions.

His work outlines the Weierstraß's form of theory of elliptic function in a complete way, by outlining the various results substantially in 'chronological' order.

In addition, he outlines many historical information, enabling the reader a complete and rigorous insight not only in the theory by itself, but in the way, it has been developed by the various mathematicians.

The Bibliography mentioned throughout the work by Enneper is – at its publishing date – a practically complete one.

It enables a rigorous outline of the historical path of the theory of the elliptic functions.

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(16) I refer the reader to my mentioned work *'Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil III ('Der bei der Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin behaltene Schwarz-Weierstrasssche Briefwechsel und die bei dem Archive der Universität zu Göttingen behaltene ausgewählte Weierstrasssche Korrespondenz')'.*

(17) See Daniels [333a].

(18) See Daniels [333a], P. 177.

(19) The greatest part of the Schwarz's manuscripts is held by the Berlin-Brandenburgische Akademie der Wissenschaften in Berlin.

(20) See Daniels [333b], P. 256.

(21) See Pringsheim [191], S. 133.

(22) See Enneper [222].

The conceptual structure of the Enneper work differs slightly from the one of Volume V of the Weierstraß's Mathematical Works, and one can consider his outlines a 'short' version, that enables the reader to know the distinct ways Weierstraß presented his results during his scientific lifetime.

Felix Müller writes in the Introduction(23)

'The peculiarity of the Enneper's work … is that the theory of the elliptic functions is outlined in his historical development.

… .

The work has been noticeably extended by introducing an outline of the Weierstraß's Theory of the elliptic functions.

… .

Further to the work by Schwarz(24), I based the completion of the Enneper work(25) on the Weierstraß's mathematical methods and results outlined in revised notes to the Lectures held by my honorable Professor at the Berlin University to which I assisted in Winter-Semester 1864-1865, in Summer-Semester 1865 and in the Winter-Semester 1866-1867(26).

The Sections are §§ 5, 6, 10, 15, 19, 23, 24, 25, 28, 29, 34, 35, 55, 56(27), which outlines the elements of the Weierstraß's theory itself, the §§ 10, 15, 23, 55 outlining the Weierstraß's Theory only.

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(23) See Enneper [222], S. V-VII.

(24) See Schwarz [II].

(25) Faber refers to the first edition of the work by Enneper.

(26) Concerning the today's fate of the Notes to the mentioned Lessons see work *Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil I ('Die im Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin beibehaltenen Akten der Kommission für die Herausgabe der Werke von Weierstrass und das heutige Schicksal der die an der Universität Berlin von Weierstrass angehaltenen Vorlesungen betreffenden Manuskripte'),* S. 126-127 respectively.

(27) The content of these Section is the following one. *§5 Bibliographical Notes on the reduction, the Weierstraß's normal form and the general Jacobi's Theorem (Richelot, Weierstraß, Gauß, the reduction of the elliptic differential to the Weierstraß's normal form*

$$\frac{ds}{\sqrt{4s^3 - g_2 s - g_3}},$$

and the Jacobi's general transformation's Theorem). §6. The first kind elliptic integral, the introduction of the elliptic functions, the differential equations for the function $sn\,u, cn\,u, dn\,u$, *and the transition from the function* $sn\,u$ *to the function* $\wp u$. *(First kind elliptic integrals, amplitude, modulus, complementary modulus, the numerical values of the function* $\mathcal{F}(\varphi, k)$, *the complete integrals* $\mathcal{K}$ *and* $\mathcal{K}'$, *the series development of* $\mathcal{K}$ *in powers of* k, *the Jacobian elliptic functions, the Gudemann's notation, the Halphen's geometric introduction of the elliptic functions, the differential equation for the three Jacobian elliptic functions, the transition from the function* $sn\,u$ *to the function* $\wp u$ *the series development of the function* $\wp u$ *in powers of* u, *and the connectins between* k *and the invariants* g_2, g_3). *§10. The Addition Theorem for the function* $\wp u$. *Imaginary argument. Periods for the function* $\wp u$ *(The Weierstraß's way to determine the formula for* $\wp(u+v)$, *the value of* $\wp(iu)$, *the introduction of the periods* 2ω *and* $2\omega'$, *and the increase of the argument by a half-period. §15. The function* σu *(The introduction of the function* σu *by Weierstraß, the series expansion, the function*

$$\frac{\sigma' u}{\sigma u},$$

The Addition Theorem. The convergence of the series expansion for the function σu, *the periodicity of the function* σu, *the representation by a double infinite product, the representation by a simple infinite product by means of the sinus-function, the expression of the function*

$$\frac{\sigma' u}{\sigma u}$$

by a sum and the proof of the formula

$$\eta\omega' - \omega\eta' = \frac{\pi}{2} i$$

§19. The Jacobi's Fundamental Theorem for the ϑ*-functions (Algebraic theorems, the Jacobi's Fundamental Theorem for the product of any of the four* ϑ*-functions and the analogous Weierstraß's three-terms* σ*-formula). §23. The special* σ*-functions and their connection with the* ϑ*-functions (The Weierstraß's introduction of the* σ*-functions*

$$\sigma_1 u, \sigma_2 u, \sigma_3 u,$$

the proof that the quotient of any two of the σ*-functions* $\sigma, \sigma_1, \sigma_2, \sigma_3$ *is an elliptic function, the representation of the special* σ*-functions by infinite products, the formulae connecting the four* σ*-functions with the four* ϑ*-functions, the partial differential equation for the function* σu, *and the comparison of the Weierstraß's theory with the Jacobi's and Abel's one). §24. Outline of the – at application's sake – most important infinite series and products of the theory of the elliptic functions (The product-formulae for the functions*

$$sn\left(\frac{2\mathcal{K}x}{\pi}\right), \ldots$$

and for the functions

$$\sigma\left(\frac{2\omega x}{\pi}\right), \sigma_1\left(\frac{2\omega x}{\pi}\right), \ldots,$$

I introduced some formulae outlined by Weierstraß's work [II].

Furthermore, I extended the original work by outlining a bibliographical-historical outline of the application of the methods of the theory of functions of one complex variable(28).

Namely, the way to investigate the elliptic integrals with complex extrema outlined by Schlömilch(29)*, …, and the study of the infinite product and infinite series development of the elliptic functions*(30).

I adapted the original content of the Notes III, IV, IX-XII of the first edition, the §47, that concerns the power series development of the elliptic functions(31)*, has been revised.*

The original Note II 'On the series of Lagrange and Fourier' has been cancelled, for the new results in order that the methods to rigorously define the Fourier series expansion of a given function underwent such a sophisticated evolution, that their application became unfitted to the goals of the work in question.

The original Note I has been enlarged by outlining the new researches on the reduction of the hyperelliptic integrals, and I added to the former Notes VI-VIII (Notes II-IV of the present edition(32)*), which concern geometric problems, two other Notes.*

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particular values, the formulae for the products of the functions $\vartheta(x), \vartheta_1(x), \dots$, consequences of the outlined results and formulae, formulae for the sum of the ϑ-functions, the elliptic functions and their squares, results determined by differentiating the outlined formulae, formulae for zero-value of the arguments the q-series and the Hermite's φ-functions). §25. The Addition Theorem of the elliptic functions and of the first kind (The Addition Theorem for the functions $sn\, u$, $cn\, u$, $dn\, u$, the Addition Theorem of the first kind elliptic integrals, the Euler's method to integrate the elliptic differential equation, the Lagrange's method to integrate the elliptic differential equation, application of the outlined results to the function $\wp u$, the Sturm's method to integrate the elliptic differential equation and historical-bibliographical notes to the Addition Theorem). §28. The second kind elliptic integrals (The ellipse's arc and the second kind elliptic integral, the complete second kind elliptic integral and its series expansion in powers of k, the Legendre's Addition Theorem for the function $E(\varphi)$, the Jacobi's trigonometric series development of the function

$$E\left(am\frac{2\mathcal{K}x}{\pi}\right),$$

the Jacobi's function $Z(u)$, formulae for the functions $E(u)$ and $Z(u)$ by means of the function $\sigma_3 u$, and the Jacobi's determination of the Addition Theorem by means of the ϑ-functions). §29. The differential equations of the first and second kind elliptic integrals referred to the modulus (The second order differential equation of $E(\varphi)$ and $Z(\varphi)$ with respect to k, the periodicity modulus of the first and second kind elliptic integrals in Weierstraß's normal form as functions of the absolute invariants $\frac{g_2^3}{g_3^2}$, the Legendre's formula

$$\mathcal{K}E' + \mathcal{K}'E - \mathcal{K}\mathcal{K}' = \frac{\pi}{2},$$

and the proof that a similar formula for the uncomplete [unvollständige] *first and second kind integrals leads to third kind elliptic integrals). §34. The Jacobi's series expansion of the third kind elliptic integrals and his first form of the Addition Theorem (The Jacobi's normal form of the third kind elliptic integrals, a series development of the third kind elliptic integrals, the Jacobi's proof of the Addition Theorem, the function $\Omega(u)$, formulae for the functions $\Pi(u, a)$ and $\Omega(u)$ by means of the function $\sigma_3 u$, and the Weierstraß's normal form of a third kind elliptic integral). §35. The properties of the third kind elliptic integrals provided by the Fundamental Theorem on the ϑ-functions (The Legendre's form of the complete third kind elliptic integral by means of first and second kind elliptic integrals, the Jacobi's Theorem on the exchange of the argument and parameter, the analogous Theorem in the Weierstraß's theory, the Addition Theorem of the third kind elliptic integrals deduced by means of the ϑ-functions, and the Jacobi's Addition Theorem of the parameter). §55. The Weierstraß's theory of the transformation of the elliptic functions (Solution of the equation*

$$\mathfrak{G}[\wp(u; g_2, g_3), \wp(u; \gamma_2, \gamma_3)] = 0,$$

The reduction of the general transformation problem '$\wp_1 u$ rational function of $\wp u$', the periods of the transformed function $\wp_1 u$, and the form of the transformed function). §56. The invariant equations, the transformed σ-function, the multiplication, historical-bibliographical notes (The form of the transformed Θ-function, the term

$$\mathcal{G}_1 = \sum \wp\left(\frac{2\alpha\omega}{n}\right),$$

the formulae connecting the invariants of the original function $\wp u$ and the invariants of the transformed function $\wp_1 u$, the cases

$$n = 1,2,3,$$

the multiplication of the function $\wp u$, the function $\frac{\sigma(nu)}{\sigma^{nn} u}$ and the differential equation for it, and historical-bibliographical notes to the Weierstraß's transformation).

(28) I observe that Weierstraß pointed out that the application of the results of the theory of functions of one complex variable is not the most suited one in order to develop the Theory of Elliptic Functions.

(29) *See Schlömilch [645], S. 374ff.* Schlömilch studied the elliptic integrals leading to the Jacobian elliptic functions only. It is self-evident that one can apply the Schlömilch's methods in order to study the elliptic integrals leading to the Weierstraß $\wp$-function. I will devote a Section of my work to accomplish this task.

(30) *See Schlömilch [645], S. 409ff.* Schlömilch studied the Jacobian elliptic functions only.

(31) This Sections deals with Jacobian elliptic functions only.

(32) Namely, Note II concerns *the Fagnano's Theorem and Landen's Theorem on the arc of an ellipse and of a hyperbola (The Fagnano's Theorem on the difference of ellipse' arcs, and it proof published in the Journal 'Produzioni di Matematiche', the representation of ellipse's and*

Note V is a bibliographical-historical outline of the new results on the geometric application of the elliptic functions[33]*, and Note VI is a bibliographical-historical outline on the connections between the theory of the elliptic functions and higher arithmetics and algebra*[34]*."*

There are many doctoral Theses based directly on the Weierstraß's Lectures, like, e.g., Kiepert[35], Simon[36], Müller[37], Gravelius[38] and Biermann[39].

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hyperbola's arcs by means of elliptic integrals and the Landen's geometric Theorems), Note III is an historical outline of the researches on the geometric applications of the elliptic integrals by Maclaurin, d'Alembert, Jacob und Johann Bernoulli, Fagnano, Euler, Legendre, and other works on Fagnano. Brinkley, Wallace, Talbot, Libri (The elastic curve and the related researches by Jacob Bernoulli and Maclaurin, the parabolic spiral and the cubic parabola, the Lemniscate and the related researches by Fagnano and Euler, the Euler's Addition Theorems for the three kind of elliptic integrals, the developments of the theory of the elliptic function brought by Fagnano, Euler and Legendre, and the successive works on the Fagnano's Theorem), and Note IV concerns *the geometric application of the second kind elliptic integrals (Theorem on confocal conics and the Lemniscate, and the works thereon by Graves, Mac Cullagh, Salmon, Hart, Chasles, and the surface of the three-axes ellipsoid and the works thereon by Legendre, Catalan, Plana, Jacobi, Lebesgue, Schlömilch, Malmsten).*

[33] This Note concerns *the connections between the spherical trigonometry and the theory of the elliptic functions, the curves, which arc can be represented by an elliptic integral, the Aronhold's parametric representation for higher order curves, the works by Clebsch and Harnack, the closure problem* ['Das Schliessungsproblem'] *for an inscribed and circumscribed polygon, the application of the elliptic functions to space curves, curvature and geodetic lines* ['kürzeste Linien']*, and transformations* ['Abbildung'].

[34] This Note concerns *the application of the elliptic functions to the number theory, some given noticeable number-theoretic functions, the application of the elliptic functions to the algebra, the cubic and biquadratic equation, the modular equation and the* 5*-degree equation, some algebraic consequences of the transformation theory* ['Theorie der Transformation']*, the algebra of binary forms, the partition equation* ['Teilungsgleichung']*, and observations on the application of the theory of the elliptic functions to mechanics and mathematical physics.*

[35] See Kiepert [214]. He writes on P. 2 *'Quos genus curvarum ab Ill. Serret inventum amplificavi; lectionibus enim, quas habuit vir Clarissimus Weierstraß de functionibus ellipticis, in methodum, qua numerus harum curvarum multis augetur, sum inductus.* 'Enneper writes on S. 560 of his work [222] *'We saw that the problem to find out all curves, whose arcs can be expressed by a first kind elliptic integral, on which one can perform the operations of addition, subtraction, multiplication and division, has kept the interest of many mathematicians. Legendre (See Legendre [346a], P. 590) discovered a sixth order curve, whose arcs can be expressed by the sum of a first kind elliptic integral and of a first kind Abel integral. Gudermann and Roberts discovered a class of spherical conic sections* ['eine Gattung sphärischer Kegelschnitte']*, which are fitted to solve the considered problem. Gudermann did not publish his results.* [Enneper outlines the bibliographical references of the works by Roberts. I do not quote them, for they are not correct]. *Serret … proves that the number of these curves is infinite. Kiepert outlined a generalization of these curves (See Kiepert [214]) by studying the Weierstraß's representation of all doubly periodic functions by means of* σ*-functions.*

[36] See Simon [295]. Simon writes on S. 6 '… *Erit igitur primum agendum de theoremate illo Illustrissimi viri Poncelet supra commemorati. Attamen liceat jam hoc loco monere, haec theoremata ita tantum generaliter valere, si et quatuor puncta intersecandi et quatuor puncta, in quibus tangentes communes conicarum tangunt, imaginaria sint et omnino una tangens realis duci possit. Quam quidem restrictionem ex ipsa problematis natura proficisci videbimus. Deinde disputabimus de compositione constantium ex invariantibus ambarum coniearum formatorum, aequationi tum necessariae intrantium; denique quomodo hae ipsae relationes liberae a factoribus abundantibus possint computari dilucidandum erit; atque ratione quidem quam maxime a priori conjiciente, calculationes evitante, ope formularnm ex functionum Ellipticarum theoria, quae mihi praeceptor Weierstraß ex officina sua pro nota ipsius humanitate et urbanitate praestitit; qua occasione formae canonicae*

$$4s^2 - g_2 s - g_3$$

nec non functionis σu *in applicationicus utilitas, immo vero necessitas jam tot ex exmplis cognita, satis dilucide apparebit.'.*

[37] See Müller [294]. Enneper writes on S. 483 of his work [222] *'I will show in the next pages that the general transformation problem treated in the previous paragraphs by applying the Abel's and Jacobi's method, can be elegantly solved by means of the Weierstraß function* $\wp(u)$*. These developments were first published by the Thesis by Felix Müller 'De transformatione functionum ellipticarum', Berlin 1867, based on the Weierstraß's Lectures.'*, on S. 495 *'In the mentioned Thesis, Felix Müller outlined a method enabling to define the 'multiplication formulae', i.e. the formulae expressing* $\wp(nu)$ *in terms of* $\wp(u)$ *… '* and, on P. 492 *'They* [Enneper refers to the *algebraic formulae between the invariants* g_2, g_3 *of the function* $\wp(u)$ *and the invariants* G_2, G_3 *of the transformed function*

$$\wp_1(u) = \wp(u) + \sum\nolimits'_{\mu,\nu} \left[\wp(u - w_{\lambda,\mu}) - \wp(w_{\lambda,\mu})\right]\Big]$$

correspond – from some points of view – to the previously studied Jacobi's modular equations. These invariants equations have been published for the first time by Felix Müller in his mentioned Thesis under Weierstraß's inducement.'.

[38] See Gravelius [543b].

[39] See Biermann [248]. He writes in the Introduction *'Cum propositum mihi sit problemata quaedam mechanica tractare, quae functionum ellipticarum sola ope penitus possint perspici, hac occasione utar, ut ostendam, quantum methodi Ill. Weierstraß, praeceptoris carissimi, valeant ad facilitatem solutionum et elegantiam formularum finalium. De quibus methodis cum nihil adhuc in publicum prodierit, unde satis cognoscatur earum indoles, de theoria pauca afferam necesse esse videtur, antequam ipsa problemata aggrediar. Sed cum ipsum ill. meum praeceptorem brevi theoriam suam publici iuris facturum esse sciam, in eo liceat acquiesam, ut formulas, quae aut ipsae in sequentibus pagellis requiruntur, aut quae ad explicandas atque definiendas significationes sunt necessariae, historice, ut ita dicam, paucis verbis commemorem.'.*

Kiepert outlines in some of his works[(40)]. the basic formulae and properties of the Weierstraß elliptic functions by simply referring to the Weierstraß's Lectures.

Kiepert writes[(41)]

"Therefore, one has to outline the multiplication formulae for any given integral multiplicator.

In order to achieve this goal, I apply the functions σu and $\wp u$ introduced by Weierstraß in his Lessons on the theory of the elliptic functions, a move I feel to be authorized by the will by Weierstraß himself to publish the connected Theorems and formulae'.

Some works are based on Weierstraß's Lectures, although they are not Notes to them, like the ones by the mentioned work by Schellbach, the works by Bolza[(42)] and Falk[(43)].

There are many treatises on the theory of the Weierstraß $\wp$-function.

In year 1932 Pringsheim[(44)] published a treatise on the theory of the monodromic analytic functions, that contains an outline of the Theory of the Weierstraß Elliptic Functions that, although its form is not in strict compliance with the Weierstraß's one and he did not investigate some topics, like, e.g., the complex multiplication of the Weierstraß $\wp$-function, is a masterwork.

He writes[(45)]

"I start in this Section a comprehensive introduction to the theory of the elliptic functions based on the Weierstraß $\wp$-function and on the ultimate form of its inversion.

I outlined in my work [191] some remarks on the historical and mathematical backgrounds, particularly concerning the links between my form and the Weierstraß's form based on pure algebraic methods outlined in his work [V].

The first definition formula of $\zeta(u)$, namely

$$\zeta(u) = \frac{1}{u} + \sum_{\nu=-\infty}^{\infty} \sum\nolimits_{\mu}' \frac{u^2}{\omega_{\mu\nu}^2 (u - \omega_{\mu\nu})} = \frac{1}{u} + \sum_{\nu=-\infty}^{\infty} \sum\nolimits_{\mu}' \left(\frac{1}{u - \omega_{\mu\nu}} + \frac{u}{\omega_{\mu\nu}^2} + \frac{1}{\omega_{\mu\nu}} \right)$$

with the convergence factor

$$\left(\frac{u}{\omega_{\mu\nu}} \right)^2$$

could provide the easiest path to the foundation of the theory of the elliptic functions.

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(40) E.g., his works [175a] and [250]. The work [250] considers the question of the partition of the Lemniscate in five parts on the ground of the Weierstraß's Lectures held in Summer-Semester 1869. See my work *Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil I ('Die im Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin beibehaltenen Akten der Kommission für die Herausgabe der Werke von Weierstrass und das heutige Schicksal der die an der Universität Berlin von Weierstrass angehaltenen Vorlesungen betreffenden Manuskripte'),* S. 128. I like to point out that, as I outlined in the linked part of my work *'Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil II ('Der wichtigste Teil des vom Geheimen Staatsarchive Preußischen Kulturbesitze in Berlin-Dahlem beibehaltenen Nachlasses von Karl Wilhelm Theodor Weierstrass')',* the drafts of the works published by Kiepert have been substantially revised by Weierstraß, even if Kiepert never acknowledged it in a fair way in the published works.

(41) See Kiepert [175a], S. 21.

(42) See Bolza [478] and [478a]. He mentioned some not better identified Lectures on hyperelliptic functions by Weierstraß and the Lectures held by Weierstraß at the University of Berlin in the Winter-Semester 1881-1882. See my work *Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil I ('Die im Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin beibehaltenen Akten der Kommission für die Herausgabe der Werke von Weierstrass und das heutige Schicksal der die an der Universität Berlin von Weierstrass angehaltenen Vorlesungen betreffenden Manuskripte'),* S. 154-156.

(43) See Falk [547b]. He refers to the Weierstraß's Lectures on the theory of the elliptic functions held in Summer-Semester 1885. See my work *Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil I ('Die im Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin beibehaltenen Akten der Kommission für die Herausgabe der Werke von Weierstrass und das heutige Schicksal der die an der Universität Berlin von Weierstrass angehaltenen Vorlesungen betreffenden Manuskripte'),* S. 160-161.

(44) See Pringsheim [98a].

(45) See Pringsheim [98a], S. 1207-1209.

A corresponding path starts with the application of the … absolute convergent series

$$\sum \frac{1}{(u-\omega_{\mu\nu})^2},$$

that, integrated, leads to the [definition] *formula*

$$\wp(u) = \frac{1}{z^2} + \sum_{\nu=-\infty}^{\infty} \sum\nolimits'_{\mu} \left[\frac{1}{(u-\omega_{\mu\nu})^2} - \frac{1}{\omega_{\mu\nu}^2}\right] = \frac{1}{u^2} + \sum_{\nu=-\infty}^{\infty} \sum\nolimits'_{\mu} \left[\frac{1}{(u+\omega_{\mu\nu})^2} - \frac{1}{\omega_{\mu\nu}^2}\right]$$

… .

Therefore, one regards the function $\sigma(z)$ *with the simple zeros*

$$2\mu\omega + 2\nu\omega'$$

as the starting point of the Theory of Elliptic Functions, instead of the function

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$$

with the simple poles

$$2\mu\omega + 2\nu\omega',$$

as effectively done by Schwarz in the work [II] by Weierstraß.'

This Pringsheim's work will be extensively outlined in my work in the Section devoted to the outline of the Pringsheim's form of the Theory of the Weierstraß $\wp$-function.

In addition, Pringsheim(46) published a work, that can be considered as a 'succinct' introduction to the theory of the Weierstraß $\wp$-function with an exhaustive and interesting historical part on the introduction of the Weierstraß $\wp$-function and with an interesting account of the reasons of the choice of the notation $\wp$.

Some parts of this Pringsheim's work are translated in the Section of my work devoted to the outline Pringsheim's form of the Theory of the Weierstraß $\wp$-function

However, the part of this work devoted to the historical analysis of the introduction of the Weierstraß $\wp$-function will be outlined later in this Chapter, for he considers the connections between the Weierstraß's theory of the elliptic functions and the Eisenstein's one.

I observe that the wife of Pringsheim burned the unpublished manuscripts after his death in year 1941.

The circumstances of these events remain mysterious.

It follows that one cannot exclude that he developed further the theory of the Weierstraß $\wp$-function.

I observe that Schwarz outlined in his Lectures a form of the theory of the elliptic functions slightly distinct from the Weierstraß's one.

An important remark on the Schwarz's form has to be done.

The correspondence between Weierstraß and Schwarz I published(47) show that Weierstraß made some remarks concerning some 'flaws' in the Schwarz's form.

However, I cannot give a complete rigorous account of the mathematical objections made by Weierstraß, for some basic letters thereon sent by Weierstraß to Schwarz are missing.

(46) See Pringsheim [191].

(47) I refer the reader to my above-mentioned work *‚Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil III (‚Der bei der Archive der Berlin-Brandenburgischen Akademie der Wissenschaften zu Berlin behaltene Schwarz-Weierstrasssche Briefwechsel und die bei dem Archive der Universität zu Göttingen behaltene ausgewählte Weierstrasssche Korrespondenz')'.*

I do not devote a Section to the Schwarz's form, for the lack of published papers thereon does not enable a rigorous analysis.

I will outline some results of his only.

Two main works on the Weierstraß elliptic functions are based on the Schwarz's form of the Theory of Elliptic Functions.

Namely, the work by Burkhardt(48) and the work by Hancock(49), based on the Schwarz's Lectures, that is the best treatise on the Theory of the Elliptic Functions written in English language.

Hancock writes(50)

"Although Professor Weierstraß lectured twenty-six times (from 1866 to 1885) in the University of Berlin on the Theory of Elliptic Functions including courses of Lectures on the application of these functions, no authoritative account of his work has been published, a quarter of a century having in the meanwhile elapsed.

It is therefore difficult to say in that part of the theory that bears his name what is due to him, what to other mathematicians.

I have derived considerable help in this respect from the lectures of Professor H. A. Schwarz, the results of which are published in his Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen.'.

In order to enable the reader a comparison of the two forms, I outline the English translation of parts of the mentioned Burkhardt's work and part of the Hancock's work, that is the most complete treatise on theory of the Weierstraß elliptic function based on Schwarz's form.

Hancock planned to publish two subsequent volumes, but he did not follow up with his plans.

The reasons why he did not publish these volumes are – as far as my knowledge extends – not yet known.

Nevertheless, one can speculate that the lack of complete Schwarz's results on these topics may have played an important role in Hancock's decision to not go on with the publishing of Volumes II and III.

He writes(51)

"… .

In Volume II a treatment of elliptic integrals is given.

Here much attention is paid to the work of Legendre, whom we may rightly regard as the founder of the elliptic functions, for upon his investigations were established the theories of Abel and Jacobi, and indeed, in the very form given by Legendre.

Abel in a published letter to Legendre wrote

'Si je suis assez heureux pour faire quelques découvertes, je les attribuerai à vous plutôt qu'à moi'

and Jacobi wrote as follows to the genial Legendre

> *'Quelle satisfaction pour moi que l'homme que j'admirais tant en dévorant ses écrits a bien voulu accueillir mes travaux avec une bonté si rare et si précieuse!*
>
> *Tout en manquant de paroles qui soient de dignes interprètes de mes sentiments, je n'y saurai répondre qu'en redoublant mes efforts à pousser plus loin les belles théories dont vous êtes le créateur'.*

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(48) See Burkhardt [2]. He writes on S. V-VI *"I cared to put into a unitary form the developments of the Riemann's methods. It is effectively surprising the small number of followers of the Riemann's methods. I am aware only of some remarks in the 1st edition of Carl Neumann's work [332] and in the Thomae's works [265], [263] and [169]. Even learned followers of the Riemann's basic views like Durège and Königsberger made a little use of* [Riemann's theory] *in the theory of the elliptic functions. (After the publishing of Riemann's work [241] one did recognize that he did not realize the power of its own theory, for he bounded himself to the case of real coefficients). In addition, many left questions linked to the inversion problem could be easily handled by applying the Riemann's method. On the other hand, it seems to be necessary to introduce the successive developments based either on Riemann's work, like, e.g., in the theory of modular functions, and on independent basis, namely in compliance with the Weierstraß's way. … I have been introduced to the Weierstraß's theory of the elliptic functions by H. A. Schwarz, … ."*

(49) See Hancock [137].

(50) See Hancock [137], P. IX.

(51) See Hancock [137], P. III-V.

True Fagnano[52]*, Euler*[53]*, Landen*[54]*, Lagrange*[55]*, and possibly others had discovered certain theorems which proved fundamental in the future development of the elliptic functions; but by the patient devotion of a long life to these functions, Legendre systematized an independent theory in that he reduced all integrals which contain no other irrationality than the square root of an expression of degree not higher than the fourth into three canonical forms of essentially different character.*

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(52) Hancock refers to the Fagnano's works [652] and [653]. Enneper writes on S. 514-518 of his work [222] *'The following Theorem 'One can define two arcs on the ellipse in infinite ways, so that their difference can be expressed by a line* ['Auf dem Umfang einer Ellipse lassen sich auf unendlich viele Arten zwei Bogen bestimmen, deren Differenz durch eine gerade Linie ausdrückbar ist'] *' has been called, in compliance with Legendre, Theorem of Fagnano. … . The peculiarity of the Fagnano's discovery, that was not recognized by Legendre, is that the way he proved this Theorem shows striking similarities to the methods leading to the Addition Theorem for the elliptic integrals adopted by the untoppable Euler. … . Fagnano published the results of his researches in 'Giornale de' Letterari d'Italia', Tomo XXVI (1745), Venezia MDCCXLVI, P. 266-279, first. This Journal published a series of Fagnano's works, reprinted in his work [653]. The Theorem by means of which one computes the length of elliptic, hyperbolic and cycloidal arcs is printed on P. 336 of Tome II of his work [653]. … . The 'Theorem to compute directly the length of the hyperbola arcs' is printed next. … . His notices 'Metodo per trovare nuove misure degli archi dell'iperbola equilatera' and 'Metodo per misurare gli archi di quella elisse conica, il di cui asse maggiore è medi proporzionale tra l'asse minore, e il doppio del medesimo asse minore' are printed on P. 504-536 of Tomo II of his work [653].'.*

(53) Enneper writes on S. 532-542 of his work [222] *'… Fagnano deduced that the quadrant of the Lemniscate can be algebraically partitioned in equal part, if, m being a positive integer, their number has the form*

$$2 \cdot 2^m, 3 \cdot 2^m, 5 \cdot 2^m$$

The elegant and magnificent Theorems of Fagnano pushed Euler to develop algebraically and to find out geometric forms of his Addition Theorem. … . Euler defined two arcs of the quadrant of ellipse, which sum can be represented geometrically in his work [162b]. Successive connected Euler's works are the works [162c, d], …, [162e]. … . The Euler's works [162f, g], which are, despite of their titles, of algebraic nature and seemingly based on Maclaurin's and d'Alembert's researches. … . Likewise, the Euler's works [162h, i] show remarkable analytical results. Namely, the work [162h] concerns … the Addition Theorem for the second kind elliptic integrals and its application to the comparison of arcs of an ellipse. In addition, on S. 35, the angle in the integral of an elliptic arc is considered the [integration] *variable. At the end of this note (namely, on p. 43), the Theorem 'Si capiatur*

$$s = \frac{a^2 q}{\sqrt{c^4 + (a^2 - c^2) q^2}}$$

erit differentia istarum formularum integralium semper algebraica:

$$\int \frac{\sqrt{a^4 + (a^2 - c^2) s^2}}{a\sqrt{a^2 - s^2}} ds = \int \frac{\sqrt{c^4 + (a^2 - c^2) q^2}}{c\sqrt{c^2 - q^2}} ds = \frac{(a^2 - c^2) q \sqrt{c^2 - q^2}}{c\sqrt{c^4 + (a^2 - c^2) q^2}}$$

The other notes on the 'elastic curve' is based on the older assertion 'plenior explicatio' … . The comparison of arcs of a curve lead Euler to extremely interesting and thoughtful considerations … . These researches, which differs strikingly from his previous ones, are the Euler's works [162j, k]. Euler observed already in his work [162l] that, under the substitution

$$x = \frac{mz + a}{nz + b}$$

the radicand does not show odd powers of z. By assuming that only even powers of x and y are shown by the radicand, Euler obtained, by consequently performing much easier computations, the noticeable results he outlined in his work [162m]. Namely, he extended his very first form of the Addition Theorem so that a theorem embodying the Legendre's Addition Theorems for the elliptic integrals of all kind. … . The mentioned Euler's works are, together with the works on the Addition Theorem, all the Euler's works on the elliptic integrals. … . One recognizes that the general form of the Addition Theorem has been outlined by Euler first and that he [based his researches] *on forms of the elliptic integrals of the three kinds, forms, which differ only slightly from the ones on which Legendre* [based his researches].

(54) Hancock refers to the work [653a] by Landen. Enneper writes on S. 523-524 of his work [222] *'This remarkable result* [Enneper refers to the Theorem that any arc of hyperbola can be expressed by means of two ellipse's arcs] *induced his discoverer Landen to state in his work [653a] 'Thus beyond my expectation, I find that the hyperbola may in general be rectified by means of two ellipses!.'. Landen announced this discovery in his work [653b] in year 1771. This notice, that concerns the difference between two arcs of an ellipse and the correspondent segments of the asymptotes of a hyperbola, shows the announcement of the considered Theorem. It has to be observed that Landen gives more weight to this Theorem, that carried not much weight in the Legendre's theory, has been considered by Landen much more important than the other paper's results, which have been nonetheless later considered extremely valuable and which has been given his name.'.* I observe that Enneper closed this Section with this interesting remark *'It is in my opinion not strictly necessary outline a complete list of the successive works dealing with these topics, for the rigorous proof of and further development of the Fagnano's Theorem faces substantial hurdles yet.'.*

(55) I observe that an exhaustive outline of the role played by the Lagrange's results has been published by Enneper in his work [222]. Enneper writes on S. III of his work [222] *'Almost any Theorem and any formula of the Theory of the Elliptic Functions is strictly connected with those three mathematicians* [Enneper refers to Legendre, Abel and Jacobi]. In addition, Enneper writes on S. 188 of his work [222] that the Euler's researches on the basic differential equation

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

are the development of the Lagrange's researches.

Thus, he was enabled to discover many of their most important properties and to overcome great difficulties, which with the means then at hand appear almost insurmountable.

Methods were devised which furnished immediate results and which, extended by subsequent investigations, enriched the science of mathematics and the fields of knowledge.

In this direction, the great English mathematician Cayley has done much work, and to him a considerable portion of this volume is due.

The admirable work of Greenhill(56) *has also been of great assistance.*

Much space is given in Volume II to the applications of the theory.

These applications are usually in the form of integrals and the results required are real quantities, and for the most part the variables must be taken real.

Thus, the complex variable of Volume I must be limited to some extent in the second volume.

The problems selected serve to illustrate the different phases treated in the previous theory; sometimes preference, as the occasion warrants, is given to Legendre s formulas, sometimes to those of Weierstraß.

While the most of these problems are taken from geometry, physics, and mechanics, there are some which have to do with algebra and the theory of numbers.

All true students of applied mathematics, engineers, and physicists should have some knowledge of elliptic functions; at the same time, it must be recognized that one cannot do all things, and it is not expected that such students should be as well versed in the theoretical side of this subject as are pure mathematicians.

For this reason, Volume II has been so prepared that without dwelling too long upon the intrinsic meaning of the subject, one may obtain a practical idea of the formulas.

Much of the theory of Volume I is therefore not presupposed, and many of the results that have hitherto been derived are again deduced in Volume II by other methods, which, without emphasizing the theoretical significance, are often more direct.

This is especially true of the Addition-Theorems.

A table of elliptic integrals of the first and second kinds will be found at the end of this volume, which may consequently, for the reasons stated, be regarded as an advanced calculus.

Volume III will be of interest especially to the lovers of pure Mathematics. In this volume, the theory becomes more abstract. Many problems of higher algebra occur which lie within the realms of general arithmetic.

This includes the theories of complex multiplication; of the division and transformation of the elliptic functions; a study of the modular equations and the solution of the algebraic equation of the fifth degree, etc.

The discoveries of Kronecker(57) *in the theory of the complex multiplication not only prove the theorems left in fragmentary form by Abel and give a clear insight into them, but they show the close relationship of this theory with algebra and the theory of numbers.*

The problem of division resolves itself into the solution of algebraic equations, and the introduction of the roots of these equations into the ordinary realm of rationality forms a 'realm of algebraic numbers'.

The same is true of the modular equations.

Kronecker, Dedekind, Hermite, Weber, Joubert, Brioschi, and other mathematicians have developed these lines of thought into an independent branch of mathematics which in its further growth is susceptible of extension in many directions, notably to the treatment of the Abelian transcendents on the one hand and of the modular systems on the other. Jacobi in a letter to Crelle wrote

> *'You see the theory* [of elliptic functions] *is a vast subject of research, which in the course of its development embraces almost all algebra, the theory of definite integrals, and the science of numbers'.*

It is also true that when a discovery is made in any one of these fields the domains of the others are also thereby extended.".

(56) Hancock refers presumably to the works [180], [393], [395], [396] and [572a] by Greenhill. I outline some parts of the Greenhill's researches later in my work.
(57) See Kronecker [88a, b, e, f].

The other best treatises have been written by Haentzschel(58), Fricke(59), Klein(60), Klein and Fricke(61), Baker(62), Hurwitz(63), Tannery and Molk(64), Appell and La Cour(65), Durège(66), Durège and Maurer(67), Frobenius and Stickelberger(68), Stieltjes(69), Goursat(70), Jordan(71), Bellacchi(72), that introduced the Weierstraß $\wp$-function in a very interesting way, Briot and Bouquet(73), Halphèn(74), Whittaker and Watson(75), Boehm(76), Weber(77), Jahnke(78),

(58) See Haentzschel [5]. He introduced the Weierstraß $\wp$-function by means of the theory of the potential equation, by developing the method introduced by Wangerin in his work [466a] and [466b]. Safford further developed the researches of Haentzschel in his work [80] and published some observations on the Wangerin's work in his work [553]. I outline the Haentschel's results in the Section of my work devoted to the form of the theory of Weierstraß elliptic functions I define the Haentzschel-Wangerin's form.

(59) See Fricke [557], [568], [568a] and [568b]. Fricke defines the Weierstraß $\wp$-function as a particular case of elliptic functions of the first rank ('elliptische Funktionen erster Stufe') in his work [568]. He introduces the Weierstraß $\wp$-function as the inversion of an elliptic integral, by briefly outlining the fundamental well-known connected formulae in his work [568a], and he outlines the theory of the elliptic functions under the provisions of the theory of the automorphic functions in his work [568b].

(60) See Klein [167], [550], [334a], [334d]. The work [167] by Klein outlines a form of the theory of the Weierstraß $\wp$-function I define 'First Klein's form', that I outline in my work. Fricke writes on p. 247 of his work [568a] '*The Weierstraß's form has a preferred room compared with the Jacobi's one in a systematic Theory of Elliptic Functions. Klein outlined the connections between these theories (See Klein [334a], [334d], [550] and [163a]) by means of the 'rank theory* ['Stufentheorie']. *The Klein's form enables a 'rank-ordered* ['Stufenweise angeordnet']*' representation of the considered theory, whose first bricks are the Weierstraß's and Jacobi's methods.'* See the work [588] by Hecke.

(61) See Klein and Fricke [198] and [163a]. These are the best treatises outlining the links between the theory of the automorphic functions and the Weierstraß $\wp$-function.

(62) See Baker [64a]. The first part shows a detailed study of the Weierstraß elliptic functions and of many geometrical applications of them. He writes on pages VI-VII *'The problem studied in the second part of the volume (the reduction of the theory of multiply-periodic functions to the theory of algebraic functions) was one of the life problems of Weierstrass, but, so far as I know, he did not himself publish during his lifetime anything more than several brief indications of the lines to be followed to effect a solution. The account given here is based upon a memoir in the third volume of the Gesammelte Werke, published in 1903; notwithstanding other publications dealing with the matter, as for example by Poincare and Picard, and particularly by Wirtinger, it appears to me that Weierstrass's paper is of fundamental importance, for its precision and clearness in regard to the problem in hand, and for the insight it allows into what is peculiarly Weierstrass's own point of view in the general Theory of Functions; at the same time, perhaps for this reason, some points in the course of the argument, and particularly the conclusion of it, seem, to me at least, to admit of further analysis, or to be capable of greater definiteness. In making this exposition I have therefore ventured to add such things as the explanation in §53, the limitation to a monogenic portion of the construct and the argument of §60, an examination of simple cases of curves possessing defective integrals and the argument of Chapter IX. These are doubtless capable of much improvement. But the whole matter is of singular fascination, both because of the great generality and breadth of view of the results achieved and because of the promise of development which it offers; I hope that the very obvious need for further investigation, suggested constantly throughout this part of the volume, may encourage a wider cultivation of the subject, and a more thorough study of the original papers referred to in the text, of which I have in no case attempted to give a complete reproduction, though I have endeavoured in all cases to acknowledge my obligations.'*.

(63) See Hurwitz [170] and [548a].

(64) See Tannery and Molk [201], [202], [203] and [204].

(65) See Appell and La Cour [474] and [578].

(66) See Durège [532] and [192].

(67) See Durège and Maurer [172].

(68) See Frobenius and Stickelberger [365].

(69) See Stielties [397a] and [397].

(70) See Goursat [97].

(71) See Jordan [189].

(72) See Bellacchi [246], P. 252-263.

(73) See Briot and Bouquet [195].

(74) See Halphén [7a, b, c], [446] and [449].

(75) See Whittaker and Watson [78].

(76) See Boehm [103] and [185]. He outlined the theory of the Weierstraß function by following methods different from the Weierstraß's ones. In addition, he does not quote Weierstraß at all in the Introductions to both Volumes of his work. Nevertheless, he outlined a deep analytical investigation of the principal formulae. I do not outline the English translation of his works for the mentioned Pringsheim works are 'de facto' an 'upgraded version' of the Boehm's works. However, I only apply some results and some formulae outlined .in the Boehm's works.

(77) See Weber [188] and [83].

(78) See Jahnke [140a].

Greenhill[79], Safford[80], Daniels[81], Forsyth[82], E. T. Bell[83], Krause[84], Krause and Naetsch[85], Pincherle[86], Bianchi[87], Peano[88], Sansone[89], Tricomi[90], Fiske[91], Söderblom[92] and Kneser[93].

Thomae[94] published in year 1905 a work claiming to be the completion and evolution of the Schwarz's work [II]. However, these claims are, in my opinion, a little bit exaggerated.

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(79) See Greenhill [572a]. He writes on P. X *'We begin with Abel's idea of the inversion of Legendre's elliptic integral of the first kind, and employ Jacobi's notation, with Gudermann's abbreviation, for a considerable extent at the outset. The more modern notation of Weierstrass is introduced subsequently, and used in conjunction with the preceding notation, and not to its exclusion; as it will be found that sometimes one notation and sometimes the other is the more suitable for the problem in hand. At the same time explanation is given of the methods by which a change from the one to the other notation can be speedily carried out.'*.

(80) See Safford [80], [553], [558], [559] and [567].

(81) See Daniels [333a], [333b] and [333c]. I observe that these works show the peculiarity that the Author applied some results outlined by Weierstraß in his Lessons held at the Berlin University and some results outlined by Schwarz in his Lessons held at the Berlin University. See the Forsyth's remarks outlined in footnote (82).

(82) See Forsyth [165] and [381]. Forsyth writes on P. 165-166 of his work [381] *'Within the last few years many investigations, founded on Weierstrass's method in the Theory of Elliptic Functions, have been published; and rather more than three years ago, there appeared the first instalment of what promises to be a full abstract of Weierstrass's Lectures on this particular subject, under the title ,Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen'. An additional reason which increases, if possible, the interest of this work is that the doubly periodic functions which enter are such as to have their periods independent of one another, instead of being functions of a single quantity as are the $4\mathcal{K}$ and $4i\mathcal{K}'$ in the Jacobian Theory of Elliptic Functions. But (as Its title implies) the work is not a full development of the theory; it is, in large part, a statement of results. After the earlier sections the results are by no means difficult to obtain; but the same remark does not apply to these earlier sections in which the most important and fundamental formulae of the theory are given. It seems desirable that some account in English should exist so as to be within convenient reach of readers, to prove sufficient to lead them to the fundamental formulae spoken of and so to enable them to follow the theory in its subsequent developments. The only account of this kind with which I am acquainted is that given by Mr. A. L. Daniels in the American Journal of Mathematics vol. VI* [Forsyth refers to the Daniels' works [333a] and [333b]] *and is contained in two notes. The second is the more important; in it he derives the results by taking as his starting-point the general theory of functions, following in this respect Weierstrass himself. In the present paper the results desired are obtained in what seems to me a slightly different way, founded on the theory of periodic functions as given in Liouville's lectures [193] and in Briot et Bouquet's work [195]. No claim for originality is here made; it is essentially a collection and reproduction, with occasional applications, of materials which will be found in the two authorities already mentioned, in the lectures of Weierstrass and in memoirs [176] by Kiepert and [358] by Simon. And nothing more than an introduction is attempted, because when the point denoted is once reached the remaining investigations can be read in, or worked out with comparative ease from, the Formeln und Lehrsätze.'*.

(83) See E. T. Bell [636] and [632a]. He studied the links between the Lucas' functions and the Weierstraß elliptic functions.

(84) See Krause [433].

(85) See Krause and Naetsch [538].

(86) See Pincherle [156].

(87) See Bianchi [161].

(88) See Peano [390]. He outlines a geometrical definition of the Weierstraß $\wp$-function.

(89) See Sansone [177].

(90) See Tricomi [624].

(91) See Fiske [414].

(92) See Söderblom [368].

(93) See Kneser [398].

(94) See Thomae [504]. He writes in the Introduction *'A collection of formulae … has been published by Schwarz* [Thomae refers to the work [II] by Schwarz]. *The great theoretical weight of this work, that is not yet completed, calls for its as soon as possible completion. This collection is substantially based on the Weierstraß's basic functions $\wp(u)$ and $\sigma(u)$. I took the decision to make available to the public a similar collection in accord with Jacobi-Legendre's notation, for, as remarked by Scheibner in his work (published in Berichte und Verhandlungen der Gesellschaft der Wissenschaften zu Leipzig, 1889), the form of these basic functions seem not to be suitable for numerical calculations. The computation of the elliptic functions with given modulus and argument is more complicated if in accord with Weierstraß's notation than if in accord with the Jacobi's ones. The function*

$$\wp(u) = -\frac{d^2\sigma(u)}{du^2} = -\frac{\sigma''(u)}{\sigma(u)} + \left(\frac{\sigma'(u)}{\sigma(u)}\right)^2$$

is not only a two-termed one, but the numerators converge remarkably slower than the Theta-functions. In addition, the Weierstraß's theory seems not to provide autonomous methods to solve the inverse problem, namely, to define the argument when the modulus and the elliptic functions are given. Schwarz fixed this problem in §48 of his work [II] by introducing the Legendre's normal form for the generally ['überhaupt'] *finite integral. Weierstraß himself based the computing of the Weierstraß's term h, that Halphèn … still denotes q, in accord with Jacobi, not on the invariants, but on the Legendre's modulus k. The assertion that the representation of the periods through g_2, g_3 by means of the hypergeometric series outlined by Bruns in his work [340]* [See footnote of my work] *provides an efficient computing tool sounds doubtful. I consider a not negligible aspect the strict analogy of the Jacobi's forms with the trigonometric functions, for it provides an easier 'well-positioning' of the formulae* ['sie dient jedenfalls zur leichteren Aneigung der Formeln'].'.

Painlevè[95] and Myrberg[96] studied the Addition Theorems for a general class of functions to develop a theory based on it.

Study[97], Pringsheim[98],Fontené[99] and Günther[100] studied the Addition Theorem for the ℘-function and some of their results will be mentioned later in my work.

Servant[101] outlined interesting results concerning the connection of the Weierstraß ℘-function and differential geometry.

Greenhill[102], Pincherle[103], von Lilienthal[104], Pick[105], Simon[106], Schwering[107], Milne and Taylor[108], Hadamard[109], Turrière[110] outlined geometrical applications of the ℘-function.

Kiepert[111] and Frobenius and Stickelberger[112] developed the complex multiplication and the transformation theory of the ℘-function.

The complex multiplication of the ℘-function has been successively studied by Hurwitz[113], Greenhill[114], Bonaventura[115], Burnside[116], Fricke[117].

Sylow[118] studied the complex multiplication of the elliptic functions in the general case – i.e. not for the Weierstraß ℘-function – in connection with the Kronecker's form of the theory of the elliptic functions.

Fricke[119], Dolbnja[120], Berwick[121] and Koschmieder[122] studied the complex multiplication of the ℘-function and the theory of the transformation of the elliptic functions, by considering the Lemniscate case too, in the 20th Century.

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(95) See Painlevè [159].
(96) See Myrberg [570].
(97) See Study [448b].
(98) See Pringsheim [178]. This work is a short outline of the main forms of the Addition Theorem of the Weierstraß ℘-function and of the Jacobian elliptic functions and of their proofs.
(99) See Fontené [457] and [468]. His researches have been considered by Hermite in his work [469].
(100) See Günther [287a] and [287b].
(101) See Servant [233a].
(102) See Greenhill [395].
(103) See Pincherle [412].
(104) See von Lilienthal [367].
(105) See Pick [382].
(106) See Simon [358].
(107) See Schwering [515].
(108) See Milne and Taylor [565].
(109) See Hadamard [580]. The works [591] and [605] by Gambier are connected to it.
(110) See Turrière [593].
(111) See Kiepert [175a], [176], [214], [357], [360], [361], [362], [369] (see the work [392] by Cayley too), [400], [407] and [416]. The works [175a], [357], [360], [361], [362], [369] and [416] by Kiepert have been in part checked and corrected by Weierstraß, and they comprehensively outline some aspects of the theory of the Weierstraß ℘-function under the structure of the Weierstraß's Lectures. The Geheimes Staatsarchiv – Preußischer Kulturbesitz in Berlin-Dahlem has a remarkable collection of letters (Ref.'s no. GStA PK, VI. HA Familienarchive und Nachlässe, Nl Weierstraß Nr. 8) sent by Kiepert to Weierstraß, where a part of the theory of the Weierstraß ℘-function is outlined. I published it with comments in the second part of my work *'Beiträge zur Geschichte der Mathematischen Werke von Karl Weierstrass. Teil II II (‚Der wichtigste Teil des vom Geheimen Staatsarchive Preußischen Kulturbesitze in Berlin-Dahlem beibehaltenen Nachlasses von Karl Wilhelm Theodor Weierstrass')'* on display on Googlebooks.
(112) See Frobenius and Stickelberger [331] and [184] and [365].
(113) See Hurwitz [500b], [500] and, concerning the Lemniscate case, [487a]. The work [494a] by Dinzl and the work [338a] by Matter concern the results outlined by Hurwitz in his work [487a].
(114) See Greenhill [393], [411], [439], [440] [470] and [572].
(115) See Bonaventura [456].
(116) See Burnside [423].
(117) See Fricke [422].
(118) See Sylow [389].
(119) See Fricke [598], [599] and [603].
(120) See Dolbnja [506].
(121) See Berwick [564].
(122) See Koschmieder [74].

An explicit 'arithmetic-topological' theory of the $\wp$-function has been developed by König and Krafft(123) in the 20th Century.

Tölke published, in the second half of the XX. Century, a collection of remarkable works(124), introducing not only the elliptic and Weierstraß elliptic functions, but also special kinds of and some functions generalizing the Weierstraß $\wp$-function. It is the best attempt ever made to generalize the theory of the Weierstraß $\wp$-function.

Some other particular results have been published by Haentzschel(125), Goursat(126), Schwarz(127), Lelieuvre(128), Hardy(129), Bruns(130), Bolza(131), Hancock(132),Igel(133), Stäckel(134), Miller(135), Lacour(136), Pellet(137), Hasse(138), Sophie von Kowalewski(139), Ernesto Pascal(140), Jahnke und Emde(141), Fueter(142), Klein(143), Painlevè(144), Study(145), Pick(146), Dolbnja(147), Falk(148), Vessot King(149), Delaunay(150), Söderblom(151), Fueter(152), Fricke(153), Kluwer(154), Witt(155), Garnier(156), Greenhill(157), Mineur(158), Mathews(159), Покровскаго(160), Takenouchi(161), Hadamard(162).

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(123) See König and Krafft [584].
(124) See Tölke [77a-g].
(125) See Haentzschel [545] and [520]. I outline some parts of the work [545], for it outlines the connections between some Euler's researches and the theory of the Weierstraß $\wp$-function.
(126) See Goursat [353a]. Goursat outlines an alternative proof to a formula for the Weierstraß $\wp$-function outlined by Halphèn and ascribed to Weierstraß, enabling to define the general integral of the Euler's equation.
(127) See Schwarz [431a]. This work, whose English translation I outline in my work, does concern the representation, involving exponential functions, of elliptic functions
(128) See Lelieuve [507].
(129) See Hardy [508]. This work, whose English translation I outline in my work, outlines a connection with the Kronecker's form of the theory of the elliptic functions.
(130) See Bruns [340]. This work outlines a method to compute the invariants g_2, g_3 and the periods of the first and second kind elliptic integrals. I outline the English translation of a great part of this work.
(131) See Bolza [478a] and [481a].
(132) See Hancock [566].
(133) See Igel [471].
(134) See Stäckel [465]. It is a basic work on the Addition Theorem for the Weierstraß $\wp$-function.
(135) See Miller [293a].
(136) See Lacour [490].
(137) See Pellet [573].
(138) See Hasse [585].
(139) See Sophie von Kowalewski [543c].
(140) See Ernesto Pascal [455].
(141) See Jahnke and Emde [140] and [140b]. They outlined some formulae and graphs for special values of the invariants g_2, g_3.
(142) See Fueter [586] and [594].
(143) See Painlevè [654]. Floquet considered some Painlevè results later in his work [486].
(144) See Klein [552a].
(145) See Study [445].
(146) See Pick [280].
(147) See Dolbnja [129], [451], [408] and [509].
(148) See Falk [514], [547a] and [544].
(149) See Vessot King [512].
(150) See Delaunay [517].
(151) See Söderblom [492].
(152) See Fueter [527].
(153) See Fricke [549].
(154) See Kluwer [482] and [495].
(155) See Witt [623].
(156) See Garnier [554].
(157) See Greenhill [180] and [572].
(158) See Mineur [577].
(159) See Mathews [556].
(160) See Покровскаго [482a].
(161) See Takenouchi [604].
(162) See Hadamard [467]. It relates to the Stieltjes's works [397a] and [397].

These works are, in general, instruments for the completion of the Weierstraß's theory of the Weierstraß elliptic functions.

I will outline some results outlined by these works as far as they give an effective contribution to the theory of the Weierstraß elliptic functions.

I could not read all the references I found(163).

However, I could read the reviews of some works published in the Jahrbuch über die Fortschritte der Mathematik(164).

As outlined, the works on theory of the $\wp$-function have been developed mainly in the 19th Century, and one can partition them into two distinct classes.

Namely, the treatises devoted to an extensive, although not complete, outline of the theory of the $\wp$-function and the works devoted to the analysis of some parts only.

Therefore, the only way to achieve the goal of this chapter is to complete the results outlined in Volumes V and VI of Weierstraß' Mathematical Works with the results outlined in works based directly or indirectly on his Lectures, and with the works on this subject by other Authors.

An aspect of the theory of the $\wp$-function is that the greatest part of the works has been written in German and, later, in French language.

However, all the French books, some translations in French language of some Weierstraß's works excepted, do not show original results.

Few works have been written in English language. Among them, the above-mentioned work by Hancock.

(163) Namely (I mention works published until year 1939 only), the work [548a] by Hurwitz, the fundamental work [447] by Tichomadritzsky, the work [427] by Anissimoff, the works [424] and [441] by Lerch the work [428] and [452] by Букреев, the works [429], [521], [529] and [539] by Гráве, the work [430] by Brodén, the works [442], [480] by Покровскаго (I observe that its second name is translated as Pokrovsky and Potocki. Under the first name the work [480a], connecting the Abel's Addition Theorem with the Weierstraß $\wp$-function (A review has been published on P. 66 of the Bulletin des Sciences Mathématiques, Deuxième Série Tome XXI Seconde Partie (1897)), and the work [480b], (A review has been published on P. 67 of the Bulletin des Sciences Mathématiques, Deuxième Série Tome XXI Seconde Partie (1897))), the works [494] and [525] by Dolbnja, the work [513] by Scheibner, the work [516] by Pech, the work [518] by Gonggryp, the works [522] and [526] by de Brun, the work [523] by Dumas, the work [528] by Falk, the works [533] and [534] by Mertens, the work [536] by Lewandowski, the work [537] by Escott, the work [543] by Hayashi, the work and [558] by Safford, the work, the work [560] by Gruder, the work [380] by Söderblom, the work [569] by Velten, the work [384] by Johansson, the work [140c] by Jahnke, the work [501] by Hardy, the work [575] by Koschmieder, the work [576] by Herglotz, the works [435], [436], [437], [438] [453] and [484] by Brioschi, the work [374] by Morera, the work [454] by Ernesto Pascal, the work [581] by Saddler, the work [541] by Moulton, the work [502] by Bauer, the work [582] by Darling, the work [587] by Skolem, the work [555] by Kneser the work [503] by Möller, the works [601] and [602] by Platone, the work [505] by Pleskot, the work [595] by Nagell, the works [589] and [596] by Hecke, the work [590] by Turrière, the work [597] by Humbert, the work [606] by Popken, the work [610] by Mahler, the work [613] by Petrovich and Karamata, the work [616] by Maier, the work [617] by Söhngen, the work [609] by Plemelj, the work [511] by Wetzler, the work [615] by von Lindemann.

(164) The work [498] by Покровскаго (Jahrbuch über die Fortschritte der Mathematik, Band 31 (1900), S. 445), that is the first part of his attempt to publish a comprehensive outline of the Weierstraß's theory, the work [409] by Dolbnja (Jahrbuch über die Fortschritte der Mathematik, Band 22 (1890), S. 449-450), that concerns the algebraic-logarithmic representation of the integral

$$\int \frac{(x+\mathcal{H})dx}{\sqrt{a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4}}$$

by means of the Weierstraß $\wp$-function, the work [413] by Dolbnja (Jahrbuch über die Forschritte der Mathematik, Band 23 (1891), S. 463-464, that concerns some integrals involving the Weierstraß $\wp$-function and the integral

$$\int \frac{x^4dx}{\sqrt{a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4}},$$

the work [364] by Stolz (Jahrbuch über die Fortschritte der Mathematik, Band 14 (1882), S. 387), that concerns basic properties of Weierstraß $\wp$-function, the work [479] by Vivanti (Jahrbuch über die Fortschritte der Mathematik, Band 18 (1886), S. 400-402), that is the transcription of a course on the subject held at the Messina University, the work [396] by Greenhill (Jahrbuch über die Fortschritte der Mathematik, Band 31 (1900), S. 443) concerning the link between cubic and quartic equations and the Weierstraß's elliptic functions, the work [496] by Mathews (Jahrbuch über die Fortschritte der Mathematik, Band 31 (1900), S. 450-451), that concerns the complex multiplication of the Weierstraß $\wp$-function in the Lemniscate case.

Weierstraß published some mathematical formulae on the σ-functions with index and on the $\wp$-function in his work [I].

Weierstraß applied some results of the theory of the $\wp$-function to prove some fundamental theorems of the theory of the analytic functions of one complex variable and of the theory of the infinite series.

I will not deeply investigate this aspect, bounding myself – if the case will be – to consider only some connected results.

Although the historical completeness calls for studying the fundamental aspect of the connection between the Weierstraß theory and the preceding Hermite theory(165), I will not perform it in this edition of my work, but I will recall, if the case can be, specific Hermite's results.

The following Klein's statements(166) outline the role played by Klein in the development of the theory of the automorphic and Weierstraß elliptic functions.

"I wrote in volume II of my Complete Works that the researches on the Icosahedra addressed me to my works on the elliptic functions, and particularly on the modular functions … for I was pursuing the understanding the methods of solution of the fifth-degree equations outlined in year 1858 by Hermite, Kronecker and Brioschi, by means of the Icosahedra theory.

… .

Therefore, I devoted myself to transformation Theory of Elliptic Functions(167), and, particularly, to a new analysis of the Hermite problem to represent in a simpler form the 5th, 7th and 11th degree resolvents, which arise, under a Galois's theorem, under 5th, 7th and 11th degree transformations.

This goal is achieved by combining some results of theory of groups and of theory of the invariants with the Riemann's theory of functions(168).

One has a complete view on the various kinds of modular equations and multiplicator equations mentioned in the literature and a deep insight into the algebraic equations solved by elliptic functions(169) as well.

These works defined a new approach the systematic treatment of all questions generated by the Theory of Elliptic Functions.

One can assert … that the infinite discontinuous groups of the homogeneous linear substitutions(170)

$$\begin{cases} u' = \pm u + m_1\omega_1 + m_2\omega_2 \\ \omega_1' = \alpha\omega_1 + \beta\omega_2 \\ \omega_2' = \gamma\omega_1 + \delta\omega_2 \end{cases} \qquad [230.\mathcal{Introduction}.I],$$

the coefficients $m_1, m_2, \alpha, \beta, \gamma, \delta$ *are given integers fulfilling the relation*

$$\alpha\gamma - \beta\delta = 1$$

play the main role, and that one has to order the subgroups in a suitable way.

The connection with the fundamental outline of Erlangen's program, i.e. that the group theory is the ordering principle in the chaos of the events, is clear. This form of the problem is correct provided two conditions are satisfied.

The first one is that I consider only finite index subgroups, so to apply the theory of algebraic functions.

The second one comes from the historical developments, which show that the subgroups mentioned by the antecedent literature were congruence groups, i.e. they were the subgroups of [the formula] *[230. Introduction. I] such that the numbers*

$$m_1, m_2, \alpha, \beta, \gamma, \delta$$

(165) See, e.g., von Caspary [391].

(166) See Klein [230a], S. 3-4.

(167) See Klein [272].

(168) See Klein [102a-b] and [551a].

(169) See Klein [334b-c].

(170) I order the formulae as follows [Bibliographical reference number (in this case 230). Chapter, section, or paragraph (in this case Introduction). Number of the formula (in this case (I))].

fulfil some well-defined congruence $\bmod n$ *conditions*(171)*.*

Therefore, the various developments can be partitioned into 1st, 2nd, 3rd, nth level.

The 1st level functions play a fundamental role in Weierstraß Theory of Elliptic Functions.

The 2nd level functions play a fundamental role in both Jacobi and Weierstraß theory, but they are unsymmetrical ones $(sn\,u\,, cn\,u\,, dn\,u)$ *in Jacobi theory, and symmetrical ones* $\left(\sqrt{\wp(u)-e_1}, \sqrt{\wp(u)-e_2}, \sqrt{\wp(u)-e_3}\right)$ *in Weierstraß's theory.*

The main features are connected by the fact that Weierstraß used u, ω_1, ω_2*, in this precise order, as homogeneous variables, as Gauß did in his researches on the arithmetic-geometrical mean, while Legendre, Abel and Jacobi operated with the ratios* $\frac{u}{\omega_2}, \frac{\omega_1}{\omega_2}$ *only.*

Until now the functions of higher level have been quite never considered in the successive developments, except in the works by Gauß and Abel and in Jacobi's researches, where they studied 2^a *and* $2^a 3$*, (a = 0, 1, …), level's functions only occasionally.*

This abstract scheme finds out its concrete outline in case of pure modular functions, i.e. of functions of

$$\omega = \frac{\omega_1}{\omega_2},$$

by means of the fundamental polygon in the ω*-plane, that enables to introduce the Riemann's existence theorems of the theory of functions of one complex variable.*

The construction of this polygon(172) *can be based on the definition of the transformed modulus of the representing systems, but the successive researches gave it a more general meaning.*

I refer the reader about the topics of Section III of Volume III to the theory of the automorphic functions or, e.g., to the main question of the modern crystallography(173)*.*

From a general point of view, my works published in this Volume, if seen under the light of the theory of invariants (i.e. of the projective way of thinking), merge the theory of groups, the numbers theory, the geometry and the theory of functions, and the method outlined therein … is the pillar of the works published in this Volume.

One can apply an important method of the progressing mathematical research, namely the algorithmic method.

By limiting myself to the works on the elliptic functions published in this Volume, I can assert that the Theta-series are applied only in a late stage and provisionally.

This fact shows some advantages, for, if I had applied the Theta-series in the initial stages, I would have not found, like some other mathematicians before me, the fundamental principle leading to the basic concepts.

… . I became aware of elliptic functions introduced by Weierstraß by means of my friend Kiepert very early (in winter 1869-70), and I came to this theory (in years 1877-78(174)*) starting from the topic of the solution of the fifth-degree equation."*

There exists a posthumous work attributed to Burkhardt and published by Faber in year 1920(175).

This work shows many variations and so new additions to the original work [2] by Burkhardt not made by the original Author.

They denaturize the original philosophy by replacing the old version of some fundamental results with the new one used in the Twenties, and it put a fence to the understanding of the fundamental concepts of the Theory of Elliptic Functions.

The Archive of the Berlin-Brandenburgische Akademie der Wissenschaften in Berlin held the handwritten corrections notes to the [II](176).

The researches I performed lead to the discovery that a great part of the considered work devoted to the σ- and $\wp$-functions is a mere copy of the Schwarz's handwritten notes and corrections.

(171) See Klein [334c].
(172) See Klein [272].
(173) See Klein [230b].
(174) See Klein [230c], S. 257.
(175) See Burkhardt [200].
(176) Ref.'s no. ABBAW Nl Schwarz 432, 434, 435.

In year 1876 Mittag-Leffler published a work, whose English translation has been published in year 1923 only(177).

In year 1926, the English translation [583] of a work by Mittag-Leffler, originally published in year 1876 in Swedish language(178), has been published.

These works are a fundamental attempt to develop the results and methods by Weierstraß under his 'philosophy'.

It is well-known that Kronecker developed in an alternative form of the Theory of Elliptic Functions. Although some authors stated that his researches started from the need to seek an application of the Theory of Elliptic Functions to the theory of numbers, and although it used distinct notations, the Kronecker theory can

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(177) See Mittag-Leffler [579]. The translator writes *'The present paper was published as a separate pamphlet in Halsingfors in March 1876 under the title „En metod att komma i analytisk besittning af de Elliptiska Funktionerna". It was presented as an academic dissertation by Professor Mittag-Leffler to the then Imperial Alexander University in Finland when competing for the chair in mathematics in that university which he subsequently held from 1877 to 1881.'. Gösta Mittag-Leffler writes 'When my paper was first published it was my intention to let it form a part of a greater memoir, comprehending a detailed exposition and critical analysis of the different methods which form an introduction to the Theory of Elliptic Functions. However, my time became fully engaged by other scientific occupations so I could never prevail upon myself to take up and carry by means of my original intention. Since I hoped to be able to carry by means of my original plan, my paper was never translated into a foreign language. I have found, however, that I cannot expect my hope to be fulfilled; however, the passing years have not seen the appearance of a satisfactory analysis of connections between Abel and Weierstraß, which, by the way, extends to many other fields than that of elliptic functions. Under these circumstances, I have accepted with sincere gratitude the offer made by Professor E. Hille to translate my memoir into English and I wish to offer him my thanks for the energy and knowledge of the subject matter which he has expended on the present paper.* [He added the following Gauß's citation *‚Ich fordere, man soll bei allem Gebrauch des Kalküls, bei allen Begriffsverwendungen sich immer der ursprünglichen Bedingungen bewußt bleiben und alle Produkte des Mechanismus niemals über die klare Befugnis hinaus als Eigentum betrachten.'*]. H*alf a century has not yet elapsed since the elliptic functions were first introduced in mathematics. From that time on the theory has increased to this extent that nowadays scarcely any other field of mathematics can offer this abundance of formal results and this wealth of applications to different branches of the exact sciences. The prophetic divination of Euler has become a reality; the discovery of this theory has essentially extended the bounds of mathematical analysis. New fields have been opened for mathematical thought and the number of fundamental ideas with which mathematics operates has been vastly increased. A careful analysis of these fundamental ideas has formed the point of departure of a great number of the investigations, the results of which form the peaks of our present-day knowledge in mathematics. This source of new and essential progress is certainly far from exhausted. However, while mathematical literature is rich in memoirs on elliptic functions, we still lack a comparative exposition and analysis of the different methods which offer an introduction to the Theory of Elliptic Functions, and of the fundamental ideas which form the starting-points of these methods and which thereby constitute them as essentially different methods. At the suggestion of Weierstraß the author of this memoir has for some time past been occupied with the solution of the problem to define by what essentially different methods an introduction to the Theory of Elliptic Functions can be obtained. The present paper forms a part of the author's investigation which can be considered to be closed in it. Its purpose is, on one hand, to expound the fundamental ideas which prepared the way and made a Theory of Elliptic Functions possible, however, to give a rigorous presentation of one of the main roads which provides an introduction to this theory. This road, the road of Abel, is historically the oldest, and it is also founded on ideas which belong to the lower layers of the mathematical structure, thus leading to the goal with a minimum number of assumptions. If the number of assumptions is increased, the method of Abel gains in unity and perfection and passes over into the older method of Weierstraß in a natural and simple manner. Thus, a description of the latter method belongs to the field of the problem which solution we aim at in the present paper. It also belongs to this field to show the deficiencies of Abel's method as well as that of Weierstraß and to denote how these can be remedied by other methods.'.*

(178) See Mittag-Leffler [583]. The translator writes *'Authorized translation from the Swedish by Einar Hille. The present paper appeared in Of versigt K. Vetenskaps - Akademiens Forhandlingar, vol. 33, No. 6 (Stockholm, 1876) with the title "En metod att i teorien for de elliptiska funktionerna harleda de oandliga dubbelprodukterna utur multiplikationsformlerna".'.* He writes *'In the memoir "En metod att komma i analytisk besittning af de elliptiska funktionerna" we have shown that the method given by Abel in his "Recherches sur les fonctions elliptiques" that is historically the earliest method of deriving the elliptic functions, is also mathematically sound and provides the desired result with perfect rigor. The method of Abel has the advantage over all other methods that it does not presuppose any theorems from the general theory of functions. It is consequently the most elementary of all mathematical processes which lead to a real command of the analytical properties of elliptic functions. A few serious objections can be raised against Abel's own presentation, however. The most important of these is that the passage from the multiplication theorems to the infinite double series and double products is not sufficiently justified; in fact, it is based upon an argument which it is scarcely possible to make mathematically rigorous. This has been pointed out by Broch in his paper "Om de elliptiske Funktioners Roekkeudvikling". In this paper, as well as in his learned and suggestive book "Traité Elementaire des Fonctions Elliptiques", there is developed a new derivation of this limiting passage that is perfectly distinct from that of Abel. In our memoir "En metod etc." mentioned above* [See Mittag-Leffler [579]] *we have later derived the infinite double series that is the general expression for Weierstraß's function $\wp(u)$, starting from the multiplication theorems. Our derivation follows immediately from the earlier work of Abel and we do not need to resort to auxiliary devices from any extraneous range of ideas. We have also denoted how the infinite double products for the differences*

$$\wp(u) - e_1, \wp(u) - e_2, \wp(u) - e_3$$

can be derived just as easily and directly. Our object in the present note shall be to show how this is to be accomplished.".

enable a remarkable development of the theory of the ℘-function as soon as a complete translation of his mathematical language into the Weierstraß one will be done.

The works [VII], [VIII], [VIIIa, b, c], [28] and [IX] represent the *summa* of the Kronecker's theory(179), and some parts of them enable to deeper explain and understand the Eisenstein's theory and its connections with the Weierstraß's one(180).

The following words by Kronecker mentioned in the letter sent by Kiepert to Weierstraß on Nov. 16, 1886(181) explain the Kronecker's way(182)

„Ich halte z. B. die ganze sogenannte analytische Theorie der ellipt[ischen] Functionen nur für etwas Formales und nur die arithmetische *Theorie derselben, namentlich die der ellipt[ischen] Functionen mit singulären Moduln für etwas* Wesentliches*.*

Mir ist es nur um Erkenntniß dieser unbegreiflich herrlichen und wunderbaren Schöpfungen eines göttlichen Geistes zu thun, und dabei verschwindet mir gänzlich das Interesse für alle die formalen Untersuchungen der modernen Literatur.

Betrachten Sie – ich habe ausdrücklich darauf hingewiesen – meine Publikationen über ellipt[ischen] Functionen nur als nothwendige Vorbereitung für diejenige, die jener Erkenntnis gewidmet sein sollen!

Ob aber überhaupt diese Publikationen vorläufig ins mathematischen Publicum dringen, ja selbst ob sie später einmal ordentlich durchdringen, das ist es nicht, was mir vor Allem am Herzen liegt.

Mir liegt nur die Erkenntniß *am Herzen, und ich bin gewiß, daß, wenn ich eine richtige Erkenntniß gewonnen und verständlich mitgetheilt habe, sie sich […] Bahn brechen wird.*

Es wird, wenn auch erst lang nach meinem Tode, ab und zu der eine oder der andere Mathematiker sich finden, der sich die echten Perlen aus den, was ich verarbeitet habe, heraussuchen wird, der ebenso wie ich und mein einziger Kenner Hermite dafür begeistert sein wird, daß ein Gott die singulären elliptischen Functionen rein arithmetisch geschaffen hat, während wir armselige Sterbliche auf dem […] analytischer Träume dazu gelangt sind.

Denn ὁδεοσαριτμητίκει hat Gauß gesagt."

The conceptual structure of the Kronecker published works(183) and the fact that the greatest part of Kronecker's notes, unpublished manuscripts and correspondence has been destroyed during WW II, do not enable

(179) Kronecker published other works on some aspects of the Theory of Elliptic Functions some tenth of years before, namely the works [88a, b, c], and he published some results of his theory in years 1881 and 1883 (See Kronecker [88d, e, f]).

(180) Hancock writes on p. V of his work [137] *'The discoveries of Kronecker in the theory of the complex multiplication not only prove the theorems left in fragmentary form by Abel and give a clear insight into them, but they show the close relationship of this theory with algebra and the theory of numbers. The problem of division resolves itself into the solution of algebraic equations, and the introduction of the roots of these equations into the ordinary realm of rationality forms a 'realm of algebraic numbers'; the same is true of the modular equations. Kronecker, Dedekind, Hermite, Weber, Joubert, Brioschi. and other mathematicians have developed these lines of thought into an independent branch of mathematics which in its further growth is susceptible of extension in many directions, notably to the treatment of the Abelian transcendents on the one hand and of the modular systems on the other. Jacobi in a letter to Crelle wrote: 'You see the theory [of elliptic functions] is a vast subject of research, which in the course of its development embraces almost all algebra, the theory of definite integrals, and the science of numbers.'. It is also true that when a discovery is made in any one of these fields the domains of the others are also thereby extended.".* He added, on p. 321, *"In his last lecture, Theorie der elliptischen Functionen zweier Paare reeller Argumente (W. S. 1891), Professor Kronecker especially emphasized the Eisenstein's theory and made paramount a certain function* $\mathcal{E}n$ *(denoting Eisenstein's name) that is a generalized ℘-function.'.* Hardy applied some results outlined by Kronecker in his work [XI] in his work [508] in order to prove some results of the Theory of Elliptic Functions.

(181) See Geheimes Staatsarchiv – Preußischer Kulturbesitz in Berlin-Dahlem, Germany, GStA PK, VI. HA Familienarchive und Nachlässe, Nl Weierstraß Nr. 8.

(182) I outline the original text because of his philosophical relevance. Kronecker writes '*I regard the whole so-called analytical Theory of Elliptic Functions as a strictly formal apparatus, and I noticeable relevant only the arithmetical theory, i.e. the Theory of Elliptic Functions with a singular modulus. I feel that such so excellent discoveries are due to the action of the Almighty, and, therefore, I have lost any interest in the formal researches of the modern literature. I refer the reader – and I did it ever explicitly – to my publications on the elliptic functions as a necessary step to achieve knowledge of such topic. The fact that the reader can directly understand these publications, or that they can be later brought into a well-organized form, does not primarily concern me. I am concerned, on the contrary, with pursuing the correct knowledge and the correct way to share it, so that it can break by means of [the audience]. Any other mathematician, who will make, also a long time after my passing away, really fine discoveries, will find that … the Almighty has created the Theory of Elliptic Functions on a pure arithmetical basis, while we, miserable mortals, are struggling with the analytic illusion, for, as Gauß said, ὁδεοσαριτμητίκει.'.*

(183) See, e.g., Kiepert [655].

to rigorously connect the Kronecker results with the Weierstraß ones, forcing me to 'neglect' temporarily the study of the Kronecker form of the theory.

However, the following above-mentioned statements published by Hancock(184)

"The discoveries of Kronecker in the theory of the complex multiplication not only prove the theorems left in fragmentary form by Abel(185) *and give a clear insight into them, but they show the close relationship of this theory with algebra and the theory of numbers.*

The problem of division resolves itself into the solution of algebraic equations, and the introduction of the roots of these equations into the ordinary realm of rationality forms a 'realm of algebraic numbers'; the same is true of the modular equations.

Kronecker, Dedekind, Hermite, Weber, Joubert, Brioschi, and other mathematicians have developed these lines of thought into an independent branch of mathematics that in its further growth is susceptible of extension in many directions, notably to the treatment of the Abelian transcendentals on the one hand and of the modular systems on the other.

Jacobi in a letter to Crelle wrote:

You see the theory [of elliptic functions] is a vast subject of research, which in the course of its development embraces almost all algebra, the theory of definite integrals, and the science of numbers. It is also true that when a discovery is made in any one of these fields the domains of the others are also thereby extended.

… ."

and(186)

"We can next consider a second form of the differential equation [562.E] by means of which the elementary elliptic functions can be defined

$$\left(\frac{dz}{du}\right)^2 = 4z^3 - g_2 z - g_3,$$

a form due to Hermite and Cayley(187).

… .

Similar results can be derived, if we write the original differential equation [562.D](188) *in the form*

$$\left(\frac{dt}{du}\right)^2 = 4t(1-t)(1-\lambda t) \quad (\mathcal{Riemann})$$

or in the form(189)

$$\left(\frac{dt}{du}\right)^2 = t(1 - \varrho t + t^2) \quad (\mathcal{Kronecker}).$$

In both cases one can introduce functions corresponding to the Theta-functions or to the σ-functions(190) *and we can define properties of these new functions properties, which are analogous to the properties of the functions just mentioned.*

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(184) See Hancock [137], P. V.

(185) Enneper writes on p. 192 of his work [222] *'Abel published the extension of the Euler's Addition Theorem to an arbitrary integral of algebraic differential in year 1825 (See Abel [642]), i.e. the well-known fundamental theorem of the theory of functions called 'Abel's theorem' by Jacobi. This theorem states that one can express the sum of an arbitrary number of functions with algebraic differentials in term of a well-defined number of such functions. This statement was published by Abel in year 1826 in his work [643] and later in his work [646].'.*

(186) See Hancock [562], P. 95 and 102.

(187) *See Cayley [314], P. 285-287 and Brioschi [625].*

(188) Hancock refers to the differential equation

$$\left(\frac{dz}{du}\right)^2 = (1 - z^2)(1 - k^2 z^2) \quad [562.D]\cdot$$

(In accord with the usual notations).

(189) *See Kronecker [28].*

(190) And, Therefore, for the $\wp$-function.

One can in this way and in a general and in a more direct manner obtain the results denoted by Riemann and Kronecker. These new functions are merely other forms of Theta-functions.

This is, of course, since the differential equations, which define the functions expressible as quotients of these new functions can be transformed into the differential equation [562.D].

It follows that that they are like the $\mathcal{Al}$*-functions of Weierstraß*(191)*, adding nothing to the general theory."*

enable a small step toward the understanding of the Kronecker form of the theory of the elliptic functions(192).

Klein connected a generalization of the double index σ-functions with some results of the Kronecker Theory of Elliptic Functions.

He writes(193)

"*The general functions* $\sigma_{x,y}(u)$ *defined by the formula*

$$\sigma_{x,y}(u\,|\omega_1,\omega_2) = e^{(x\eta_1+y\eta_2)\left(u-\frac{x\omega_1+y\omega_2}{2}\right)}\sigma(u - x\omega_1 - y\omega_2|\omega_1,\omega_2) \quad [550.13]$$

are shown in the literature in a different way, and I want only point out that the 'analytical invariant Λ*' introduced by Kronecker in his newest researches on the Theory of Elliptic Functions*(194) *is, in accord to my notations,* $\ell n\left|\sqrt[12]{\Delta}\sigma_{x,y}\right|$*."*

An attempt to connect the Kronecker's form with the theory of the Weierstraß elliptic functions has been made by Wilhelm Maier in the Thirties of the past Century(195).

The works by Eisenstein(196) deserve a separate analysis.

It is well-known that these works embody the fundamental formulae of the $\wp$-function later introduced by Weierstraß with different notations.

Pringsheim(197) outlined the links between the Eisenstein's Theory of Elliptic Functions and the Weierstraß theory of the $\wp$-function in a very elegant way.

Pringsheim writes(198)

"… .

IV. On the designation 'elliptic functions' and the inversion of the Weierstraß $\wp$*-function.*

It is common knowledge that the denomination 'elliptic functions' adopted one hundred years ago conceals a completely 'ellipse-friendly' kind of monodromic, and due to their peculiar properties, noticeable analytic functions usually roughly defined fractional ('meromorphic') doubly periodic functions(199).

This definition can be explained by considering the historical development of the concepts, which it is based on, but, nonetheless, it is not fully legitimate.

Legendre defined the whole group of integrals

(191) Weierstraß outlined the theory of the $\mathcal{Al}$-functions in his work [86].
(192) There exist some works enabling to understanding some parts of the Kronecker theory, like the work [391a] by von Caspary and the mentioned work [508] by Hardy.
(193) See Klein [550], S. 345. He applied his own notations.
(194) *See Kronecker, particularly [VIIIa, b] and Kronecker [88e, f].*
(195) See Wilhelm Maier [588a], [588b] and [589a]. He writes on page 85 *'The considered approach is, in agreement with the Weierstraß's one, based on the definition of a particular way to introduce the elliptic functions* ['Der vorliegende Ansatz geht mit Weierstraß von einer besonderen Darstellung aus, um die elliptischen Funktionen einzuführen']. *In compliance with the Kronecker's way, one renounces to absolute convergence of the limit processes. Therefore, one can freely define the representative function, by keeping it 'so flexible' that it can embody Bernoulli's polynomials and periodic functions.'.*
(196) See Eisenstein [205]-[212], [215] and [273].
(197) See Pringsheim [191].
(198) See Pringsheim [191], S. 129-135.
(199) ‚die man sinngemäßer als gebrochene („meromorphe") doppelperiodische Funktionen zu charakterisieren pflegt '.

$$\int \frac{\mathcal{P}dx}{\sqrt{\mathcal{R}}},$$

([the term] $\mathcal{P}$ *being a rational function of* x, *and* [the term] $\mathcal{R}$ *being a* 4*th degree polynomial)*

'elliptic transcendents'

at first(200)*, and*

'elliptic functions'

after(201)*, for a particular case of the integral in question shows up in the rectification of an ellipse (and, generally, of a hyperbola).*

It is hard to regard this terminology as a quite fitted one(202)*.*

However, a further noticeable worsening occurred in years 1825-1829, for Abel and Jacobi expanded – more or less contemporaneously – this research field by introducing the inverse of the Legendre first kind elliptic functions.

Jacobi seized this opportunity to call the inverse functions he introduced 'elliptic functions', the Legendre's elliptic functions being integrals' notation only.

This definition has been held ever since, despite the vigorous complaints by Legendre(203) *and the initial agreement by Jacobi to amend the terminology accordingly*(204)*.*

These facts show that the path to the well-defining of the origin of term 'elliptic' needs a further move.

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(200) *See Legendre [179].*

(201) *See Legendre [179a], [346] and [346a].*

(202) *Weierstraß writes on S. 2 of his work [III] 'These integrals bear this name because of the completely uninfluential fact that one among them represents the arc of an ellipse.'.*

(203) *Legendre wrote to Jacobi on July 16, 1829 'Je devrai borner là ma lettre et ne vous parler des changements de nomenclature que vous proposez dans votre article 17 pag. 31 (sc. Fundamenta nova); mais comme d'autres personnes pourraient vous représenter qu'en cela vous avez fait une chose qui doit m'être désagréable, je ne vois pas pourquoi je vous cacherais ce que je pense de cette proposition. Je vous dirai donc franchement que je n'approuve pas votre idée et que je ne vois pas de quelle utilité elle peut être pour vous et pour la science. … . Il me suffira de vous avoir témoigné ma surprise sur l'inconvenance et la bizarrerie de votre idée; elle n'altérera en rien les sentiments d'estime et d'affection que j'ai conçus pour vous et dont je vous renouvelle l'assurance. Le Gendre'. See Jacobi [399], S. 451.* I observe that the omitted part plays nonetheless a fundamental role in the understanding of the Legendre's objections. Namely, he writes *'La plus simple des fonctions elliptiques, savoir l'intégrale*

$$\int \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

jouit de tant et de si belles propriétés ; considéré seule, elle est liée par de si beaux rapports avec les deux autres fonctions dites de la seconde et de la troisième espèce que l'ensemble de ces trois fonctions forme un système complet auquel on pourrait donner un autre nom que celui de fonctions elliptiques, mais dont l'existence est indépendante de toute autre fonction. La nomenclature méthodique que j'ai proposée, des 1793, dans mon mémoire sur les transcendantes elliptiques, a été adoptée généralement, vous 1 avez trouvée établie; quelles sont donc vos raisons pour vous écarter de l'usage général? Vous faites schisme avec M. Abel et avec moi, vous faites schisme avec vous-même, puisque, après avoir appelé fonctions elliptiques les sinus, cosinus et autres fonctions trigonométriques de l'amplitude, vous êtes encore oblige d'appeler fonctions de troisième espèce celles que je désigne sous le même nom. N est ce pas ce que veut dire le titre de l'art. 56 p. 160? Pourquoi désignez - vous comme moi la fonction de 3e espèce tantôt par $\Pi(u,a)$*, tantôt par* $\Pi(u, a+\mathcal{K}', k')$*? Quelle liaison y a-t-il entre ces fonctions et la première, qui n'est plus, suivant vous, qu'un argument de fonction? Je vous laisse à expliquer toutes ces choses. Du reste, je vous fais part confidentiellement de ces observations, dont vous ferez tel usage que vous voudrez, et auxquelles je ne donnerai jamais aucune publicité.'.*

(204) *See Jacobi [399], S. 452.* Jacobi writes (It is in my opinion that the reader can get the point better by reading the original version instead the English one) *'Comme j'écris ceci en hâte, je ne puis répondre que quelques mots aux reproches que vous m'avez faits dans votre dernière lettre, et pour lesquels je vous rends grâce mieux encore que pour les éloges que vous m'avez prodigué et que j'ai si peu mérités. II me fallait absolument une dénomination pour les fonctions sin am, cos am, etc., dont les propriétés répondent parfaitement à celles des fonctions sin, cos, dites circulaires. D'un autre côté, l'application importante qu'on fait de la théorie des fonctions elliptiques au calcul intégral rendait nécessaires les distinctions et les dénominations que vous avez introduites dans l'analyse, et qui ont été accueillies par tous les géomètres. J'ai donc trouvé convenable d'appeler les intégrales auxquelles vous donnez le nom de fonctions elliptiques de la première, seconde, troisième espèce, intégrales elliptiques de la première, seconde, troisième espèce et d'étendre ou d'attribuer de préférence la dénomination de fonctions elliptiques aux sin am, cos am, ⊿am, analogiquement comme on nomme fonctions circulaires les sinus, cosinus, etc. Si cela vous déplait, toute autre dénomination me sera agréable. Dans tous les cas, je crois que nous deviendrons aisément d'accord sur cet objet (*[Note by Borchardt *La correspondance, interrompue après cette lettre par le voyage de Jacobi en France et par son séjour à Paris, n'a été reprise que l'année suivante et ne s élevé plus à son niveau antérieur, les fonctions elliptiques ne formant plus, ni pour Legendre ni pour Jacobi, l'occupation presque exclusive).'*

I could state only, in order to fix the origin of the definition and at the best of my will, that the term 'elliptic function' means the inverse function of an integral with a form strictly resembling, but with fundamentally distinct character, the one [of the function] *representing the arc of an ellipse.*

The sought definition seems to me one of the worst definitions which can be found in the whole analysis. ... .

This part of my 'critical-historical remarks' is only a correlated result of my vigorous attempt at the study of the historical origin of elliptic functions.

These researches provided to the following fundamental question

> *'Is the existence of the so-called 'elliptic functions', i.e. 'the fractional doubly periodic functions' originated from being they inverse functions of the first kind elliptic integral, a sufficient one in order that this connection between these two kinds of functions is the most natural one?'*

The talented ... Eisenstein[205] could have cut off the question ... in his large and outstanding and still to be fully understood work published in year 1847[206]... .

The 'double products' are formed by linear factors of the Weierstraß σ-functions without the exponential factors necessary for their continuous convergence[207].

I.e. they are only continuously convergent[208] ones, as proved by studies of the double series defined by its logarithm.

This last statement would hold in case of the Eisenstein double series corresponding to the partial fractions expansion of $\wp(u)$ without the term

$$-\frac{1}{\omega_{\mu\nu}^{2}},$$

i.e. in the case of the series

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[205] *He was born in Berlin in year 1823. The volumes 27-41 of the Crelle's Journal (1844-1852) contain more than forty his great and short notices.* The Eisenstein's works on these topics are the works [205]-[212], [215] and [273].

[206] *See Eisenstein [210].*

[207] 'unbedingte Konvergenz'. I observe that Pringsheim does not state a rigorous definition of continuous convergence of an infinite double product. Eisenstein itself does not quote the term 'continuous convergence of infinite double product' in his work in question. Namely, he writes on S. 191 of his work [210] *'The convergence of the infinite double products subject of this work, like the convergence of the connected double series, by performing the multiplication in the product and the sum in the series with respect to the index (m) at first and to the index (n) after, will be proved by performing a multiplication or a sum (with respect to* [the index] *m) by using the circular and exponential functions at first, and by investigating the convergence of the so obtained simple products or simple sums of the terms with index n.'*. It follows that Pringsheim had to have proven that the double infinite products are not only convergent, but, more specifically, conditionally convergent. Pringsheim (See Pringsheim [460], S. 134) defines a series 'continuosuly convergent', if any change in the order of its terms provides a convergent series with the same sum. The properties of continuous convergence in the finite and absolute convergence in the finite in the Pringsheim's theory of series do coincide. I observe that it does coincide substantially with the one outlined by Eisenstein in his work [210].

[208] *The conditional convergence* ['bedingt Konvergenz'] *has, in this case, a meaning differing from the one in case of the double series. See Pringsheim [460], S. 140.* Pringsheim writes on S. 140 *'On the other hand, the concept of conditional convergence, and, therefore, the one of convergence of a double series, has been often too vaguely defined, by stating that a double series is defined conditionally convergent if the series of the terms $S_{\mu}^{(\nu)}$ possesses a finite limit under a particular limiting process, e.g. for*

$$\mu=\varphi(\varrho) \quad \nu=\psi(\varrho)$$

and

$$\lim_{\varrho=\infty}\varphi(\varrho)=\infty \quad \lim_{\varrho=\infty}\psi(\varrho)=\infty$$

(see Eisenstein [210], S. 172). This allowed generalization of the double series convergence concept is not much useful, for it excludes the possibility to represent a double series by a simple symbol like

$$\sum_{\mu,\nu=0}^{\infty} a_{\mu}^{(\nu)}$$

Therefore, it does not enable to apply the fundamental analogy between simple and double series.'. Pringsheim outlines an analysis of these concepts in his work [98a]. I observe that Pringsheim writes '*A convergent double series, that is not defined by the property to be an absolute convergent series, is defined a conditionally convergent series. It has been proved that it can be transformed into a divergent series, by a mere interchange of its terms.'*. See Pringsheim [98h].

$$\frac{1}{u^2}+\sum\nolimits'_{\mu,\nu}\frac{1}{(u-\omega_{\mu\nu})^2}$$

Eisenstein, however, 'helped himself'.

He subtracted the in the same sense unconditionally convergent series

$$\sum\nolimits'_{\mu,\nu}\frac{1}{\omega_{\mu\nu}^2},$$

and he obtained the usual conditionally convergent partial fractions expansion(209). *The correspondent series for* $\wp'(u)$ *is not continuously convergent.*

---

(209) *The absolute convergence of the series*

$$\sum\nolimits'_{\mu,\nu}\frac{1}{(u-\omega_{\mu\nu})^a}$$

for

$$a>2$$

is a particular case of a correspondent theorem for multiple series proved on pp. 157ff. of his work [210]. Eisenstein writes on S. 153-157 of his work [210] *'The theory of circular functions shows infinite products, which factors are defined by the property to vanish for any value of the variables, that is element of a two-sided arithmetic succession, i.e. for any value of the form*

$$\alpha m+\beta,$$

for any positive and negative value, and for zero-value of [the term] m *from* $-\infty$ *to* ∞, [the terms] α,β *being given constants. Likewise – mutatis mutandis – the theory of the elliptic functions shows double infinite products defined by the property that the roots of the single factors are well-defined by a 'double-headed* ['mit doppeltem Eingang'] *arithmetic series, i.e. by the terms of a double series with general term*

$$\alpha m+\beta n+\gamma,$$

[the terms] m *and* n *taking independently any value from* $-\infty$ *to* ∞. *The infinite products shewn by the theory of circular functions have the form*

$$\prod\frac{x}{\alpha m+\beta}$$

and the infinite double product shewn by the theory of the elliptic functions have the form

$$\prod\left(1-\frac{x}{\alpha m+\beta n+\gamma}\right)$$

Before turning to the study of these double product, I like to investigate the term

$$\alpha m+\beta n+\gamma,$$

[the coefficients] α,β,γ *being – in the general case – complex-valued ones. The term*

$$u=\alpha m+\beta n+\gamma$$

can be considered like a form, by mean of which one can represent given values. … . One can denote the modulus of a complex number by $\mathfrak{M}$ *One can investigate the properties defining a product in the best way by studying the* [natural] *logarithm of the single factors. One assumes that the variable* x *shewn by the product*

$$\prod\left(1-\frac{x}{u}\right)$$

is a complex-valued one. The [natural] *Logarithm of the term*

$$1-\frac{x}{u}$$

can be represented by a convergent series in powers of x, *provided that* [the inequality]

$$\mathfrak{M}(u)>\mathfrak{M}(x)$$

[holds]. *It follows that it is indicated to eliminate the factors such that* [the formula]

$$\mathfrak{M}(u)\leq\mathfrak{M}(x)$$

[holds]. *The above-outlined results show that the number of such factors is a finite one, and one denotes by*

$$\prod{}'\left(1-\frac{x}{u}\right)$$

the product of the factors fulfilling [the inequality]

$$\mathfrak{M}(u)>\mathfrak{M}(x)$$

Under these assumptions, [the formula]

$$\ell n\left(1-\frac{x}{u}\right)=-\frac{x}{u}-\frac{x^2}{2u^2}-\frac{x^3}{3u^3}-\frac{x^4}{3u^4}-\cdots$$

[holds], *and, therefore, the* [natural] *logarithm of the product is* [defined by the formula]

$$\ell n\prod{}'\left(1-\frac{x}{u}\right)=-x\sum{}'\frac{1}{u}-\frac{x^2}{2}\sum{}'\frac{1}{u^2}-\frac{x^3}{3}\sum{}'\frac{1}{u^3}-\frac{x^4}{4}\sum{}'\frac{1}{u^4}\dots\quad[210.1],$$

the sums Σ' *being over the integral values of the indexes* m *and* n *such that* the inequality]

$$\mathfrak{M}(u)>\mathfrak{M}(x)$$

[holds], *and the ordering of the values of* u *in the double sum, i.e. the order, which the terms are summed in, is the same of the order, which the factors of the product are multiplied in. … . The sums*

… .

The further differentiation and successive eliminations of the [Eisenstein's] *differential equation*(210) *provides the differential equation*

$$\wp'^2(u) = \big(\wp(u) - \wp(\omega)\big)\big(\wp(u) - \wp(\omega')\big)\big(\wp(u) - \wp(\omega'')\big) = (\wp(u) - a)(\wp(u) - a')(\wp(u) - a'')$$

In addition, one proves that, if

$$\wp(u) = x,$$

the formula

$$u = \int \frac{dx}{\sqrt{(x-a)(x-a')(x-a'')}}$$

holds. In other words, one proves that the inverse function of $\wp(u)$ *for given periods* ω, ω' *appears a first kind elliptic integral, where* a, a', a'' *must be considered well-defined functions of* ω, ω'*, whereas the question of the correspondent formula for arbitrarily given* a, a', a'' *remains unsolved. Weierstraß, after discovering the function* $\wp(u)$ *thanks to the Eisenstein's researches*(211)*, recognized its fundamental importance for the theory of the 'elliptic' functions and the theory of the (fractional) doubly periodic functions that he*

$$\sum{}' \frac{1}{u^{\mu}}$$

do converge independently by the ordering of the terms, and by replacing them with their moduli, for

$$\mu > 2$$

… *. This Theorem is a particular case of a more general one, namely 'The* τ*-fold series*

$$\sum \frac{1}{(m_1^2 + m_2^2 + m_3^2 + \cdots + m_\tau^2)^{\mu}}$$

the indexes

$$m_1, m_2, m_3, \ldots, m_\tau$$

taking any integral value from $-\infty$ *to* ∞*, but the combination*

$$m_1 = 0 \quad m_2 = 0 \quad m_3 = 0 \quad \ldots \quad m_\tau = 0,$$

does converge if [the inequality]

$$\mu > \frac{1}{2}\tau$$

[holds]*.'.'*

(210) *See Eisenstein [210], S. 226.* Eisenstein writes on pages 225-226 *'In order to give to this third degree function a more elegant form, …, one can look for three values of* x *such that*

$$(3, x) = 0$$

They are

$$x = \frac{\alpha}{2} \quad x = \frac{\beta}{2} \quad x = \frac{\alpha+\beta}{2}.$$

One proves these formulae by applying the property of periodicity and the formula

$$(3, -x) = -(3, x)$$

… *. Therefore, by putting*

$$\begin{cases} (2, x) = y & \left(2, \frac{\alpha}{2}\right) = a \\ \left(2, \frac{\beta}{2}\right) = a' & \left(2, \frac{\alpha+\beta}{2}\right) = a'' \end{cases}$$

… *one obtains* [the differential equation]

$$\frac{\partial y}{\partial x} = \sqrt{2(y-a)(y-a')(y-a'')}$$

[and the formula]

$$2x = \int \frac{\partial y}{\sqrt{2(y-a)(y-a')(y-a'')}}$$

… *.".*

(211) *A remark in a footnote on P. 175 of the work [201] by Tannery and Molk sheds some light on an aspect of the disagreements occurred between Weierstraß and Kronecker in last years of their live. Kronecker would have liked to denote the considered function with the symbol* $en(u)$ *in honor to and in memory of Eisenstein instead of the symbol* $\wp(u)$.

constructed by applying the principles of the analytic functions outlined in his classical memory[(212)] *representing one of the most fecund chapters of his theory of functions.*

However, until today, a complete authentic outline of the theory that Weierstraß developed in his Lectures held at the Berlin University[(213)] *does not exist."*

Klein[(214)] writes

"*Eisenstein*[(215)] *outlined some formulae connected with the here considered normal form*

$$\int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

with the difference that the invariance of the terms g_2 *and* g_3 *was unknown to Eisenstein.*

His work is based mainly on the double periodicity.

Weierstraß based on this principle the theory leading to the discovery of the function $\sigma(u)$.*"*

Fricke[(216)] writes

"*Eisenstein got the impulse to some researches on the elliptic functions from the work by Jacobi*[(217)] *and from the discoveries of Gauß in the theory of numbers.*

Eisenstein[(218)] *studied the Jacobi algebraic approach the transformation theorem of elliptic functions in Lemniscate case thoroughly.*

The differential equation

$$\frac{dy}{\sqrt{1-y^4}} = (a+ib)\frac{dx}{\sqrt{1-x^4}} \quad [568a.163],$$

$(a+ib)$ *being an even complex integer of the norm*

$$p = a^2 + b^2,$$

must be solved by

$$y = x\frac{\mathcal{A}_0 + \mathcal{A}_1 x^4 + \cdots + \mathcal{A}_{\frac{1}{4}(p-1)} x^{p-1}}{1 + \mathcal{B}_1 x^4 + \cdots + \mathcal{B}_{\frac{1}{4}(p-1)} x^{p-1}} \quad [568a.164]$$

The connected researches lead to the arithmetic of the complex integers $(a+ib)$, *and one among the main goals is to define the reciprocity law of biquadratic rest on these grounds.*

Eisenstein inferred a proof of the reciprocity law of the cubic rest in the case of elliptic functions now defined 'the equiharmonic case' from the transformation theory in a successive work[(219)]. *In addition, he defined the Addition Theorem in a successive work*[(220)], *by differentiation and by means of the Taylor's Theorem*[(221)], *by starting from the differential equation*

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(212) *See Weierstraß [221].*
(213) *The Lectures on the Theory of Elliptic Functions published in Weierstraß's work [III] follow a distinct trend.*
(214) See Klein [167], S. 24.
(215) *See Eisenstein [210].*
(216) See Fricke [568a], S. 243-246.
(217) *See Jacobi [79].*
(218) *See Eisenstein [205].*
(219) *See Eisenstein [210].*
(220) *See Eisenstein [205].*
(221) *See Eisenstein [208].*

$$\frac{dx}{dt} = \sqrt{1 - \alpha x^2 + x^4} \quad [568a.165]$$

The Eisenstein's work [210] shows that he is the precursor of Weierstraß. It is based on the infinite product

$$\prod \left(1 - \frac{x}{\alpha m + \beta n + \gamma}\right) \quad [568a.166],$$

where (m, n) *have all pairs of integral values, and* α, β, γ *are complex constants.*

The constant γ *plays a secondary role, while* α *and* β *(the periods) are such that their quotient* $\frac{\beta}{\alpha}$ *is not real.*

In order to study the convergence, Eisenstein considered the logarithm of the product [568a.166], i.e. the series

$$\ell n \prod \left(1 - \frac{x}{\alpha m + \beta n + \gamma}\right) = -x \sum \frac{1}{u} - \frac{x^2}{2} \sum \frac{1}{u^2} - \frac{x^3}{3} \sum \frac{1}{u^3} - \cdots \quad [568a.167],$$

by introducing, for sake of shortness, the formal relation

$$u = \alpha m + \beta n + \gamma$$

In addition, he proved the continuous convergence of the double sum

$$\sum \frac{1}{u^\mu}$$

for

$$\mu \geq 3$$

whereas the first two series of [568a.167] converge only conditionally.

He proved that one defines their behaviour by changing the order of their terms, so to prove the convergence of the series [568a.167].

Eisenstein applied this result to state the initial concepts of the Weierstraß σ*-function, but to obtain the function itself, he missed to introduce the exponential factor enabling the convergence(222) in the product [568a.166].*

However, he studied the relations between the two functions proved by him to be elliptic functions and later denoted by Weierstraß with $\wp$ *and* $\wp'$ *in §5 of the mentioned work.*

In order to represent the elliptic functions by double infinite series of rational terms, Eisenstein applied a principle, which, by neglecting secondary amendments, has been applied by Poincaré(223). Eisenstein introduced the term

$$w = \alpha m + \beta n$$

and the series

$$(g, x) = \sum \frac{1}{(x + w)^g} \quad [568a.168]$$

Both series(224)

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(222) *This product, for*

$$\gamma = 0,$$

has been studied by Cayley in his work [472a] two years earlier and put as basis of the Theory of Elliptic Functions. The conditionally convergence of the product requires the law under which the integers m, n *approach infinite. Therefore, the pillar of the research is that a modification of this law implies that an exponential factor, whose exponent is a second-degree integral function of* x*, affects the value of the product.*

(223) *See the Poincare's theory of automorphic functions and Hurwitz [170], Chapter II.*

(224) *The star means that one has to neglect the combination*

$$m = 0, n = 0$$

$$(2, x) - (2^*, 0) \quad (3, x),$$

provide substantially the functions ℘ and ℘′, and, by suitably applying some elementary algebraic relations, one proves the formula

$$\begin{array}{c}(3, x)^2 = [(2, x) - (2^*, 0)]^3 - 15(4^*, 0)[(2, x) - (2^*, 0)] + \\ +10[c - (2^*, 0)(4^*, 0)]\end{array} \quad [568a.169],$$

i.e.[225]

$$(3, x)^2 = \left[(2, x) - \left(2, \frac{\alpha}{2}\right)\right]\left[(2, x) - \left(2, \frac{\beta}{2}\right)\right]\left[(2, x) - \left(2, \frac{\alpha+\beta}{2}\right)\right] \quad [568a.170]$$

By applying the formula [568a.168], one proves the formula

$$\frac{d(g, x)}{dx} = -g(g + 1, x) \quad [568a.171],$$

that provides the special formula

$$\frac{d[(2, x) - (2^*, 0)]}{dx} = -2(3, x) \quad [568a.172]$$

By introducing the formal relations

$$\begin{cases} y(x) = (2, x) - (2^*, 0) \\ y\left(\frac{\alpha}{2}\right) = a \\ y\left(\frac{\beta}{2}\right) = a' \\ y\left(\frac{\alpha+\beta}{2}\right) = a'', \end{cases}$$

[by applying] *the formula [568a.170],* [one] *proves the formula*

$$\left(\frac{dy}{dx}\right)^2 = 4(y - a)(y - a')(y - a'') \quad [568a.173]$$

It defines the connection with the first kind elliptic integral."

and[226]

"*Eisenstein*[227] *studied the representation of infinite series of doubly periodic functions, and one easily recognizes the invariance of the represented terms, under linear substitutions of the periods.*

The Eisenstein's computations were really 'near' such invariance, although its fundamental role was not recognized."

I observe that Schwering[228] published three interesting works on the 'Teilungsgleichung' of the Lemniscate function, by outlining interesting connection with the Eisenstein's and Kronecker's theory.

I observe that the analysis of the published works on the ℘-function shows that there are distinct ways to develop its theory all differing from the Weierstraß original one, like the Aronhold-Clebsch's, the Klein's and the Haentzschel-Wangerin's way.

in the summation sign.

(225) The next results have been published by Eisenstein on pages 225-226 of his work [210] in a slightly distinct form.

(226) See Fricke [568b], S. 354.

(227) *See Eisenstein [210].*

(228) See Schwering [238a], [238b] and [238c].

The Aronhold-Clebsch's way starts from the theory of ternary cubic forms and third order curves(229).

Study(230) published a succinct, but comprehensive, outline thereof.

Klein(231) extended these results and applied the theory of elliptic modular functions to the theory of the ℘-function.

Hurwitz(232) published further developments.

The works by Morera(233), Pick(234), Molien(235), Biedermann(236), Kiepert(237), Fricke(238), Weber(239), Jouël(240), Fiedler(241) Bolza(242), Seiffert(243) are based on or connected to the Aronhold-Clebsch-Klein's theory.

The theory of the Weierstraß ℘-function from the point of view of the Klein's theory shows some links with the Kronecker's Theory of Elliptic Functions and the same comments made about the works by Eisenstein hold in case of Klein's works.

One has to quote the fundamental work by Riemann(244), where the notes added by Stahl show the path from the Riemann's theory to the Weierstraß's one.

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(229) See Clebsch [283], S. 631-660. I consider this method the most complete 'geometrical method' providing the definition of the ℘-function and of the Jacobian elliptic functions. The Weierstraß's original way is the easiest one, but the Clebsch's way enables a direct connection with some other quantities, like the Hessian, and, implicitly, with the analytic and, if one looks for further generalizations, with the absolute differential geometry. See Pick [383].
(230) See Study [448a] and [448]. This work is a continuation of the Study's work [448b].
(231) See Klein [167] and [249].
(232) See Hurwitz [235], [500a] and [487b].
(233) See Morera [372] and [373].
(234) See Pick [375] and [377].
(235) See Molien [378].
(236) See Biedermann [379] and [394].
(237) See Kiepert [400] and [407].
(238) See Fricke [401].
(239) See Weber [403], [404], [220] and [473] (based on some letters by Cayley. See Cayley [472]).
(240) See Jouël [459].
(241) See Fiedler [370].
(242) See Bolza [481].
(243) See Seiffert [456a]. I could not read the work, but only an extended review published on P. 337-340 in the Jahrbuch über die Fortschritte der Mathematik, Band 27 (1896).
(244) See Riemann [241].